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EXTENSION OF THE BERRY-ESSEEN ESTIMATE TO CONVERGING SEQUENCES OF ABSORBED RANDOM WALKS

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled DISTRIBUTION FUNCTIONS OF A CONVERGING SEQUENCE OF MARKOV PROCESSES submitted by JARED CHAPIN in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

The distribution function at a fixed time of a random walk discrete Markov process is compared to that of an associated continuous process. The difference is shown to be proportional to the square root of the time increment of the discrete process.

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I wish to thank my supervisors, Dr. Geoff Butler and Dr. John Gamlen for their help with this thesis. Where possible their contributions are itemized below. Dr. Gamlen recognized the possibilities of the "operational calculus approach" given in Chapter Two and improved upon the arguments given there. He wrote the appendix and Section Two of Chapter One. Also he is responsible for the proofs of Lemmas Three and Thirteen in Chapter Three. Dr. Butler has been very helpful with discussions, advice and diligent checking of technical proofs. He is responsible for the proof of crucial Lemma One in Chapter Three.

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CHAPTER ONE

ESTENSION OF THE BERRY-ESSEEN ESTIMATE

TO CONVERGING SEQUENCES OF ABSORBED RANDOM WALKS

The intent of this thesis is to add to the understanding of the rate of convergence of discrete Markov processes to a continuous process. The main result addresses the fixed time distribution functions of convergent discrete Moran type processes.

The first section of this chapter gives definitions and background for some questions of convergence. The second section introduces the "boundary value problem" approach to problems of convergence. In the third section some previous results are cited. Finally the fourth section states the main result, which will be proved in Chapters Two and Three.

Section One

In this section Markovian transition functions are defined. The results of the thesis are all stated and proved within this context. Following the definition the connection with a Markov process regarded as a stochastic process is explained.

The section concludes with some examples illustrating convergence problems associated with a sequence of random walks.

Definition

Let x be a measurable subset of R, and S a subset of

$[0,\infty)$ which contains 0 and is closed under addition. Denote by $\mathcal{B}$ the Borel σ-algebra of X. A function $\{ P_t(x,U) | t \in S,\ x \in X,\ U \in \mathcal{B}\}$ is called a transition function on the state space X if:

1) $P_t(x,\cdot)$ defines a probability measure over $\mathcal{B}$.

2) $P_0(x,\{x\}) = 1$.

3) $P_t(\cdot, U)$ is $\mathcal{B}$ measurable, for fixed $U \in \mathcal{B}$.

4) $P_\bullet(\cdot,\cdot)$ satisfies the Markov identity:

$$P_{t+s}(x, U) = \int P_t(y,U)\, P_s(x,dy) \quad .$$

It is possible to construct a family $\{Z_t^x \,|\, x \in X,\ t \geq 0\}$ of random variables on the state space X such that for each $x \in X$, $\{Z_t^x \,|\, t \geq 0\}$ is a (Markov) stochastic process whose starting random variable Z_0^x is almost surely the constant random variable x. We refer to this as "the process which starts at x". These processes are Markov Processes in the classical sense (cf. 5 Ch. 3). Their joint distributions are uniquely determined by their distributions, given the Markov property. These distributions are given by the transition function as follows:

$$\text{Prob } (Z_t^x \in U) = P_t(x,U) , \quad t \geq 0, \quad x \in X .$$

Before further reduction in generality is considered, two classes (discrete and continuous) of transition functions will be identified. A transition function $P_\bullet(\cdot,\cdot)$ will be called discrete if S and X are discrete. A transition function P will be called continuous if $S = [0,\infty)$, X is connected and the following limits exist for $\delta > 0$:

$$\text{(o)} \quad \lim_{t \to 0} \frac{1}{t} \int_{|y-x|>\delta} P_t(x,dy) = 0 \quad ,$$

$$\text{(i)} \quad \lim_{t \to 0} \frac{1}{t} \int_{|y-x|<\delta} (y-x)\, P_t(x,dy) = b(x) \quad ,$$

$$\text{(ii)} \quad \lim_{t \to 0} \frac{1}{t} \int_{|y-x|<\delta} (y-x)^2 P_t(x,dy) = 2a(x) \quad .$$

The quantities $b(x)$ and $2a(x)$ may be interpreted as "infinitesimal mean and variance". Kolmogoroff observed that, if a continuous transition function $P_{\cdot}(\cdot,\cdot)$ has a continuous second derivative at $x \in X$, then it satisfies the "backward equation",

$$\frac{\partial}{\partial t} P_t(x,U) = a(x) \frac{\partial^2}{\partial x^2} P_t(x,U) + b(x) \frac{\partial}{\partial x} P_t(x,U)$$

for $t > 0$

(cf. [11], p. 299)

Situations occur where a notion of convergence of discrete transition functions to a continous transition function is useful. Perhaps the best known example occurs with symmetric random walk, described next. Let n be fixed,

let $X^n = \{ \frac{j}{n} : j$ an integer $\}$.

Let $S^n = \{0, \tau, 2\tau, \ldots\}$ where $\tau = \frac{1}{n^2}$.

$$\text{Let} \quad P^n_\tau(x,\{y\}) = \begin{cases} \frac{1}{2} & \text{if} \quad |y-x| = \frac{1}{n} \\ 0 & \text{otherwise} \end{cases} .$$

Note that the last expression together with the Markov identity completely defines $P_t^n(x,U)$ for all $t \in S^n$. As $n \to \infty$ (for those n where $x \in X^n$ and $t \in S^n$), application of the central limit theorem yields that the distributions of $P_t^n(x,\cdot)$ converge to the Normal distribution:

$$P_t(x,U) = \int_U \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t}\, dy \quad .$$

Note that this choice of $\tau = 1/n^2$ causes the variance of the n^{th} distribution at time t to be t, which is the right kind of normalization for the central limit theorem.

Also the Berry-Esseen theorem gives the rate of covergence as $1/n\sqrt{t}$ (cf. [2], p. 206). It is this rate that one expects to find in related covergence problems.

Observe that, even though "infinitesimal moments" do not exist for discrete $P^n_\cdot(\cdot,\cdot)$, "approximate infinitesimal moments" exist as indicated below:

$$2a_n(x) = \frac{1}{\tau} \sum (y-x)^2\, P_\tau^n(x,\{y\}) = 1,$$

$$b_n(x) = \frac{1}{\tau} \sum (y-x)\, P_\tau^n(x,\{y\}) = 0 \quad .$$

In the above expressions, integration over a discrete measure becomes a sum. Now it comes as no surprise that the limit transition function $P_\cdot(\cdot,\cdot)$ satisfies the backward equation:

$$\frac{\partial}{\partial t} P_t(x,U) = \frac{1}{2} \frac{\partial}{\partial x^2} P_t(x,U) \quad .$$

Now another example, the Wright-Fisher process from population genetics.

Let $X^n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$.

Let $S^n = \{0, \tau, 2\tau, \ldots\}$ where $\tau = \frac{1}{n}$.

Let $P^n_\tau(x,\{y\}) = \binom{n}{\ell} x^\ell (1-x)^{n-\ell}$ where $y = \frac{\ell}{n}$.

(cf. [3], p. 371)

As before, the one step probability defines $P^n_t(x,U)$ for all $t \in S^n$. The combinational nature of $P^n_t(x,U)$ makes it difficult to analyze for large n using the Markov identity. However, the "approximate infinitesimal moments" can be calculated:

$$2a_n(x) = \frac{1}{\tau} \sum (y-x)^2 P^n_\tau(x,\{y\}) = x(1-x),$$

$$b_n(x) = \frac{1}{\tau} \sum (y-x) P^n_\tau(x,\{y\}) = 0 .$$

Then if the right kind of convergence applies $P^n_\bullet(\cdot,\cdot)$ may be analyzed by considering solutions to the backward equation

$$\frac{\partial}{\partial t} P_t(x,U) = \frac{x(1-x)}{2} \frac{\partial}{\partial x^2} P_t(x,U) .$$

In this case the time increment τ has no meaning in the model giving rise to $P^n_\bullet(\cdot,\cdot)$. Here τ was chosen so that "approximate infinitesimal moments" converge.

Two types of situation where this "

serve mention here.

The above examples are of the first type. Suppose a discrete

transition function is defined by its first step transition probabilities. Attempts at analyzing its behavior after many steps by using the Markov identity may fail, due to the complexity of this one step transition probability. However it may be well behaved in the sense that, the transition function may be associated with an element of a sequence where the "approximate infinitesimal moments" converge. Thus analysis of the original transition function may be carried out by analyzing the associated backward equation.

Another type of situation is the reverse of that previously described. Suppose a numerical solution to a backward equation is desired. One possibility is to invent a sequence of discrete transition functions, with the first step transition function being simple (as in random walk), and the "approximate infinitesimal moments" converging to the respective coefficients of the backward equation. If the right kind of convergence exists, the solution to the backward equation may be approximated by choosing a discrete transition function far along in the sequence, and then using the Markov identity.

It may happen that the sequence of processes converges in a reasonable sense, and yet the limiting process is not the unique diffusion with the limiting infinitesimal moments. Consider this example.

Let $X^n = \{\ell/n : \ell \text{ an integer}\}$.

Let $S^n = \{0, \tau, 2\tau, \ldots\}$ where $\tau = 1/n^2$.

$$\text{Let } P^n_\tau(x,\{y\}) = \begin{cases} 1/2n^2 & \text{if } |y-x| = 1 \\ 1 - 1/n^2 & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

Observe that $2a_n(x) = 1$ and $b_n(x) = 0$.

The limiting process is a spatially homogeneous jump process, and is certainly not the standard Wiener process. Note that property (o) of page 2 fails for this process.

It is natural to ask the following question. Suppose, a continuous transition function $P_\cdot(\cdot,\cdot)$ satisfies a backward equation. Suppose, a sequence of discrete transition functions $P^n_\cdot(\cdot,\cdot)$ is given. When, in what sense, and how fast does $P^n_\cdot(\cdot,\cdot)$ converge to $P_\cdot(\cdot,\cdot)$? In this thesis, a special case of this question is examined.

Section Two

Let $P_\cdot(\cdot,\cdot)$ be a transition function as defined in the previous section. Following W. Feller and others, consider for a (temporarily fixed) bounded Borel measurable function f the expression:

$$\psi(x,t) = \int f(y)\, P_t(x,dy) \ , \quad x \in [0,1], \quad t \geq 0 \ .$$

(This is the expected value of $f(X_t)$ in the process which starts at x.) Note that if f is χ_U (the indicator function of a Borel set U), then $\psi(x,t) = P_t(x,U)$. If the "diffusion conditions" (o), (i), (ii) are satisfied it follows by a standard argument (cf. [11], p. 299) that for $f \in C^2[0,1]$,

$$\frac{\partial\psi}{\partial t}(x,\,0^+) = b(x)f'(x) + a(x)f''(x) \;, \qquad \text{a.e.}$$

Because $P_0(x,\cdot)$ is δ_x (the point mass at x), it follows from (*) that $\psi(x,0) = f(x)$. If the endpoints $0,\ 1$ are assumed to be absorbing then $P_t(0,\cdot) = \delta_0$, and $P_t(1,\cdot) = \delta_1$, all t. Equivalently, the probability of moving away from 0 or away from 1 is zero. This forces $\psi(0,t) = f(0)$, $\psi(1,t) = f(1)$, all t. These properties characterize the transition function, as noted in the appendix. A precise version of the boundary value problem satisfied by $\psi(\cdot,\cdot)$ is stated shortly.

A crucial problem is to show the existence of $P_{\cdot}(\cdot,\cdot)$ having given drift and variance velocities $b(\cdot)$, $a(\cdot)$, and absorbing ends. This thesis needs to use not only the existence, but an explicit formula for $P_{\cdot}(\cdot,\cdot)$, in the case when $b(\cdot) = 0$, and $a(\cdot)$ is Riemann integrable with $0 < k_1 \leq a(\cdot) \leq k_2$. The existence in a general setting is a very deep result due to Stroock and Varadhan(cf. [13], Chapter 3). The appendix proves existence in the particular case above, by solving a boundary value problem using eigenfunction transforms. The connection is made with the semigroup point of view, and the diffusion properties (o), (i), (ii) are proved to hold for the constructed transition function.

The BVP (boundary value problem) satisfied by $P_t(x,U)$ is now made precise when $U = [0,z]$. If $\psi(x,t)$ is defined by $\psi(x,t) = P_t(x,U)$, then:

1) $\psi(0,t) = 1 \;,\quad \psi(1,t) = 0 \;,\quad t \geq 0 \;;$

2) $$\lim_{t\to 0^+} \psi(x,t) = \begin{cases} 1 & x < z \\ 0 & x > z \end{cases}$$

3) $\frac{\partial \psi}{\partial x}$ is absolutely continuous in x on $[0,1]$ for $t > 0$,

4) $a(\cdot) \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial \psi}{\partial t}$ for almost all x, $t > 0$.

A discrete boundary value problem version of a discrete transition function is obtained simply by changing notation. First, consideration of discrete transition functions $P^n_{\cdot}(\cdot,\cdot)$ will be restricted to those with time set $S^n = \{0, \tau, 2\tau, \ldots\}$ for $\tau > 0$.

Let $\psi^n(x,t) = \int f(y) \, P^n_t(x,dy)$.

Then $P^n_{\cdot}(\cdot,\cdot)$ is uniquely identified by the solution to the boundary value problems

$$\frac{\Delta\psi^n(\cdot,t)}{\tau} = \frac{\psi^n(\cdot,t+\tau)-\psi^n(\cdot,\ t)}{\tau} = A^n\psi^n(\cdot,t) \quad ,$$

$$\psi^n(x,0) = f(x), \quad \psi^n(0,f) = f(0), \quad \psi^n(1,t) = f(1) \text{ for } x \in X^n, \quad t \in S^n.$$

Here

$$A^n g(x) = \frac{1}{\tau}\left(\sum_{y \in X^n} g(y) \, P^n_\tau(x,\{y\}) - g(x) \right) .$$

Questions of $P^n_t(x,U)$ converging to $P_t(x,U)$ may be attacked by considering convergence of A^n to A.

Section Three

In this thesis the type of convergence considered will be convergence of distribution functions at a fixed time. Thus define $P^n_{\cdot}(\cdot,\cdot) \to P_{\cdot}(\cdot,\cdot)$ if the following two conditions hold:

1) for $x \in X$, $t \in S$, there are sequences x_n, t_n with $x_n \in X^n$, $t_n \in S^n$, $x_n \to x$ and $t_n \to t$,

2) for x, x_n, t and t_n as above,

$$\int \chi_{[0,z]}(y)\,(P^n_{t_n}(x_n,dy) - P_t(x,dy)) \to 0 \ .$$

An early result of this type is due to P. Lax (cf. [9]) his convergence result verified numerical methods for solving boundary value problems.

More recently, semi-group techniques developed by Ethier and Norman (cf. [10]) yield the following result.

Let $\left| \frac{1}{\tau} \int (y-x) \quad P^n_\tau(x,dy) - b(\tau) \right| < K_1 \tau$.

Let $\left| \frac{1}{\tau} \int (y-x)^2 \ P^n_\tau(x,dy) - 2a(x) \right| < K_2 \tau$.

Let $\frac{1}{\tau} \int |y-x|^3 \ P^n_\tau(x,dy) \quad < K_3 \ \tau$.

Then

$$\left| \int f(y)\,(P^n_t(x,dy) - P_t(x,dy) \right|) < K_4 \ \|f\|_{C_4} \ \tau \ ,$$

$$\text{for} \quad x \in X^n, \quad t = k\tau, \quad \text{and some} \quad K_4 > 0.$$

This last result can yield a rate of convergence for fixed time distribution functions, by using differentiable approximations to $\chi_{[0,z]}(x)$. However the rate of convergence obtained this way is slower than that proved here.

Section Four

In this section the main result of this thesis is precisely stated. The proof of this result occupies Chapters Two and Three. In less formal terms, the result is an estimate of the difference between the fixed time

distributions of a discrete and continuous process. The continuous process is a diffusion with zero drift and absorbing ends. It would be desirable to hold this continuous process fixed while it is compared to a sequence of discrete processes. However, it simplifies the technical details if the infinitesimal variance is taken as a step function approximation to a continuous function. The discrete process is a random walk process with zero mean and "approximate infinitesimal variance" equivalent to that of the continuous process.

<u>Theorem</u>: (Main Result)

Suppose $a(x) > \varepsilon > 0$ has bounded variation and is continuous on $(0,1)$.

Define X^n, $a^n(x)$, τ, S^n, $P_t^n(x,U)$, $\psi(x,t)$ and $P_t(x,U)$ by the expressions which follow:

Let n be an integer.

Let $X^n = \{x_j : x_j = \frac{j}{n}, \ j \text{ an integer}, \ 0 \leq j \leq n\}$.

Let $a^n(x) = a\left(\frac{x_{j-1} + x_j}{2}\right)$ if $x_{j-1} < x \leq x_j$, $0 \neq x_j \neq 1$.

Let $a^n(0) = a^n(1) = 0$.

Let $\tau = \dfrac{1}{4\|a^n\|_\infty n^2}$.

Let $S^n = \{0, \tau, 2\tau, \ldots\}$.

Let the discrete transition function

$$\{P_t^n(x,U) : t \in S^n, \ x \in X^n, \ U \text{ a Borel set in } X^n\}$$

be defined by

$$P^n_\tau(x,\{y\}) = \begin{cases} \dfrac{a^n(x)}{4\|a^n\|_\infty} & \text{if } |y-x| = \dfrac{1}{n} \\ 1 - \dfrac{a^n(x)}{2\|a^n\|_\infty} & \text{if } y = x \\ 0 & \text{otherwise} \end{cases} .$$

Define $\psi(x,t)$, dependent on $Z \in [0,1)$, to be the solution to

$$\frac{\partial}{\partial t}\psi(x,t) = a^n(x)\frac{\partial^2}{\partial x^2}\psi(x,t) \quad \text{a.e. for } x \in (0,1),\ 0 < t ;$$

$$\psi(x,0) = \chi_{[0,z]}(x) \quad \text{for all } x \in [0,1] ,$$

$$\psi(0,t) = 1 \quad \text{and} \quad \psi(1,t) = 0 \quad \text{for all } t \geq 0.$$

Define the continuous transition function

$$\{P_t(x,U) : 0 \leq t,\ x \in [0,1],\ U \text{ a borel set in } [0,1]\}$$

by

$$\{P_t(x,[0,z]) = \psi(x,t) : z \in [0,1)\} .$$

Then there is a constant $K > 0$ independent of n, x and z such that

$$|P^n_t(x[0,z]) - P_t(x,[0,z])| \leq K/n$$

for those n where $t \in S^n$ and $x,z \in X^n$. K is of the form $a + b/\sqrt{t}$.

CHAPTER TWO

It is the purpose of this thesis to prove

$$\left|\int_0^1 \chi_{[0,z]}(y)P_t(x,dy) - \int_0^1 \chi_{[0,z]}(y)P_t^n(x,dy)\right| \le K/n$$

where K is a constant independent of n. There is nothing to prove if $z = 1$. In the remainder of the thesis assume $z \in [0,1)$.

In section one of this chapter the above integrals to be compared will be expressed as solutions to continuous and discrete boundary value problems respectively. In section two, it will be shown that with a condition, the solutions to these boundary value problems can be represented as contour integrals in the complex plane. Finally in Chapter Three it will be shown that the condition for the contour integral solutions are met, and these integrals will be compared to give the final result.

Section One

Let $\psi(x,t) = \int_0^1 \chi_{[0,z]}(y)P_t(x,dy)$. Recall from Chapter One that $\psi(x,t)$ satisfies the boundary value problem

$$a^n(x)\,\frac{\partial^2\psi(x,t)}{\partial x^2} = \frac{\partial\psi(x,t)}{\partial t}\ ;$$

$$\psi(0,t) = 1, \quad \psi(1,t) = 0, \quad \psi(x,0) = \chi_{[0,z]}(x).$$

The above differential equation holds where $a^n(x)$ is differentiable and $t > 0$.

This boundary value problem can be transformed into a zero end condition problem as follows. Let

$$\phi(x,t) = \psi(x,t) - (1-x)$$

(i.e. subtract the equilibrium solution to get zero end conditions). Then $\phi(x,t)$ satisfies

$$a^n(x) \frac{\partial^2 \phi(x,t)}{\partial x^2} = \frac{\partial \phi(x,t)}{\partial t} ;$$

$$\phi(0,t) = \phi(1,t) = 0, \quad \phi(x,0) = f(x).$$

In the above boundary value problem the restrictions on x and t are the same as before and

$$f(x) = \chi_{[0,z]}(x) - (1-x).$$

This is the continuous boundary value problem which will be analyzed later.

The next task is to develop an equivalent notion for the discrete boundary value problem.

A direct calculation will be done to show that

$$a^n(x)\,\frac{\Delta_x^2}{h^2}\,\psi^n(x-h,t) = \frac{\Delta_t}{\tau}\,\psi^n(x,t);$$

$$\psi^n(0,t) = 1, \quad \psi^n(1,t) = 0,$$

$$\psi^n(x,0) = \chi_{[0,z]}(x).$$

Here $$\Delta_x\, f(x,t) = f(x+h,t) - f(x,t) \;,$$

$$\Delta_t\, f(x,t) = f(x,t+\tau) - f(x,t) \;,$$

$$h = 1/n \;.$$

Recall from Chapter One

$$P_\tau^n(x,\{y\}) = \begin{cases} \dfrac{a^n(x)}{4\|a^n\|_\infty} & \text{if } |y-x| = h \\[2ex] 1 - \dfrac{a^n(x)}{2\|a^n\|_\infty} & \text{if } y = x \\[2ex] 0 & \text{otherwise,} \end{cases}$$

$$\tau = \frac{1}{4n^2\|a^n\|_\infty} \;.$$

Observe

$$\int_0^1 \chi_{[0,z]}(y)\,P_t^n\,(x,dy) = \sum_{j=0}^{n} \chi_{[0,z]}(y_j)\,P_t^n(x,\{y_j\})$$

where $y_j = j/n = jh$.

Now, to do the direct calculation use the Markov property to get:

$$\psi^n(x,t+\tau) = \sum_{j=0}^{n} \chi_{[0,z]}(y_j) P^n_{t+\tau}(x,\{y_j\})$$

$$= \sum_{\ell=0}^{n} \sum_{j=0}^{n} \chi_{[0,z]}(y_j) P^n_\tau(x,\{y_\ell\}) P^n_t(y_\ell,\{y_j\})$$

$$= \frac{a^n(x)}{4\|a^n\|} \sum_{j=0}^{n} \chi_{[0,z]}(y_j) P^n_t(x-h,\{y_j\})$$

$$+ \left(1 - \frac{a^n(x)}{2\|a^n\|_\infty}\right) \sum_{j=0}^{n} \chi_{[0,z]}(y_j) P^n_t(x,\{y_j\})$$

$$+ \frac{a^n(x)}{4\|a^n\|_\infty} \sum_{j=0}^{n} \chi_{[0,z]}(y_j) P^n_t(x+h,\{y_j\}).$$

Thus

$$\frac{\Delta_t}{\tau} \psi^n(x,t) = \frac{1}{\tau} \frac{a^n(x)}{4\|a^n\|}$$

$$\times \left[\sum_{j=0}^{n} \chi_{[0,z]}(y_j) P^n_t(x-h,\{y_j\}) - 2 \sum_{j=0}^{n} \chi_{[0,z]}(y_j) P^n_t(x,\{y_j\}) \right.$$

$$\left. + \sum_{j=0}^{n} \chi_{[0,z]}(y_j) P^n_t(x+h,\{y_j\}) \right]$$

$$= a^n(x) \frac{\Delta_x^2}{h^2} \psi^n(x-h,t).$$

Since $a^n(0) = a^n(1) = 0$, observe

$$\psi^n(0,t) = 1 \quad \text{and} \quad \psi^n(1,t) = 0.$$

Also direct calculation gives

$$\psi^n(x,0) = \chi_{[0,z]}(x).$$

As in the previous case, let $\phi^n(x,t) = \psi^n(x,t) - (1-x)$. Then

$$a^n(x)\,\frac{\Delta_x^2}{h^2}\,\phi^n(x-h,t) = \frac{\Delta_t}{\tau}\,\phi^n(x,t),$$

$$\phi^n(0,t) = \phi^n(1,t) = 0,$$

$$\phi^n(x,0) = f(x),$$

where $f(x) = \begin{cases} x & \text{if } x \le z \\ x-1 & \text{if } z < x. \end{cases}$

This is the discrete boundary value problem which will be analyzed later.

Section Two

The results of this section are

$$\phi(x,t) = \oint e^{\lambda t} R_\lambda f(x)\,d\lambda$$

and

$$\phi^n(x,t) = \oint (1+\lambda\tau)^k R_\lambda^n f(x)\,d\lambda, \quad \text{where} \quad t = k\tau .$$

The first integral is a special case extension of operational calculus to an unbounded spectrum.

The following solution to the continuous boundary value problem uses Sturm Liouville theory which is summarized in the appendix, and theory of integrals of Banach space valued functions (cf. [4], Chapter 3).

Define an inner product on $L^2[0,1]$ given by

$$\langle Y_1, Y_2 \rangle = \int_0^1 Y_1(x) \frac{1}{a^n(x)} \overline{Y}_2(x)\, dx.$$

Let A be the operator on a subspace of $L^2[0,1]$ defined by

$$AY(x) = a^n(x) \frac{d^2 Y(x)}{dx^2} .$$

Note that A is self adjoint on the approprate domain [see appendix] with respect to the inner product defined above and boundary conditions. Let λ_j and Y_j be the eigenvalues and normalized eigenfunctions of A respectively. Then the solution to the continuous boundary value problem $\phi(x,t)$ is given by

$$\phi(\cdot,t) \approx \sum_{j=1}^{\infty} e^{\lambda_j t} \langle f, Y_j \rangle Y_j(\cdot),$$

[see appendix]

where "$\approx$" in the above expression and those that follow means equal in L^2.

Cauchy's theorem yields

$$\phi(\cdot,t) \approx \sum_{j=1}^{\infty} \frac{1}{2\pi i} \oint e^{\lambda t} \frac{\langle f, Y_j \rangle Y_j(\cdot)}{\lambda - \lambda_j}\, d\lambda$$

where each contour is around the single eigenvalue λ_j.

Observe that each eigenvalue is on the negative real axis [see appendix]. The contour, given by

$$\lambda = (1-r^2) - i2r \quad \text{for} \quad -\infty < r < \infty$$

(so that $\sqrt{-\lambda} = r + i$), forms a parabola around the negative real axis.

The exponential damping effect allows the replacement of each small contour with the open parabolic contour described above. Thus,

$$\phi(\cdot,t) \approx \sum_{j=1}^{\infty} \frac{1}{2\pi i} \oint \frac{e^{\lambda t} \langle f, Y_j \rangle Y_j(\cdot)\, d\lambda}{\lambda - \lambda_j} .$$

Next the dominated convergence theorem for Banach space valued integrals will be used to show

$$\phi(\cdot,t) \approx \frac{1}{2\pi i} \oint e^{\lambda t} \sum_{j=1}^{\infty} \frac{\langle f, Y_j \rangle Y_j(\cdot)}{\lambda - \lambda_j}\, d\lambda .$$

The above statement is valid because the partial sums are L_2 dominated by the real valued integrable function $e^{\lambda t}\|f\|_2$, and $|\lambda - \lambda_j| \geq 1$.

Recall

$$\sum_{j=1}^{\infty} \frac{\langle f, Y_j \rangle Y_j(\cdot)}{\lambda - \lambda_j} \approx R_\lambda f(\cdot) = (\lambda - A)^{-1} .$$

[see appendix]

Thus

$$\phi(\cdot,t) \approx \frac{1}{2\pi i}\oint e^{\lambda t} R_\lambda f(\cdot)\,d\lambda.$$

It will be shown that $\phi(x,t) = \frac{1}{2\pi i}\oint e^{\lambda t} R_\lambda f(x)\,d\lambda$ for $t > 0$.

Consider the expression

$$\oint e^{\lambda t} R_\lambda f(x)\,d\lambda.$$

Suppose $R_\lambda f(x)$ is dominated by a polynomial in λ uniformly for $x \in [0,1]$. Then the above integral exists for each x. Furthermore, note that $R_\lambda f(x)$ is uniformly continuous in x and λ, for $x \in [0,1]$ and λ in a compact subset of the contour. Again the dominated convergence theorem yields that the integral with value in $C_o[0,1]$ given by

$$\oint e^{\lambda t} R_\lambda f(\cdot)\,d\lambda$$

exists and

$$\left(\oint e^{\lambda t} R_\lambda f(\cdot)\,d\lambda\right)(x) = \oint e^{\lambda t} R_\lambda f(x)\,d\lambda,$$

since the linear injection of $C_o[0,1]$ into $L^2[0,1]$ is continuous. The L^2 and C_o versions of the integrals are equal almost everywhere. So

$$\phi(x,t) = \frac{1}{2\pi i}\oint e^{\lambda t} R_\lambda f(x)\,d\lambda \qquad \text{a.e.}$$

Finally since both sides of the previous equality are continuous, the a.e. may be dropped. Thus

$$\phi(x,t) = \frac{1}{2\pi i}\oint e^{\lambda t} R_\lambda f(x)\,d\lambda.$$

Contour Integrals in the Discrete Case

Define A^n by

$$A^n\phi^n(x,t) = a^n(x)\,\frac{\Delta_x^2}{h^2}\,\phi^n(x-h,t).$$

Then a direct calculation will be done to show that the solution to

$$A^n\phi^n(x,t) = \frac{\Delta_t}{\tau}\,\phi^n(x,t)$$

is

$$\phi^n(x,t) = (1+\tau A^n)^k\phi^n(x,0),$$

where $t = k\tau$, k is a non-negative integer, and $x = jh$ for $0 < j < n$. Observe that if ϕ^n is expressed as above, then

$$\frac{\Delta_t}{\tau}\,\phi^n(x,t) = (1+\tau A^n)^k\frac{[(1+\tau A^n)-1]}{\tau}\phi^n(x,0).$$

Thus $\frac{\Delta_t}{\tau}\phi^n(x,t) = A^n\phi(x,t)$ as required. The $(n-1)$-dimensional vector space

$$\left\{\begin{matrix} g:\ g(x_j) & \text{is a complex number,} & 0 \leq j \leq n, \\ & g(0) = g(1) = 0 & \end{matrix}\right\}$$

can be given an inner product $\langle\, ,\rangle$ as follows,

$$\langle g_1, g_2\rangle = \sum_{j=1}^{n-1} g_1(x_j)\,\frac{1}{a^n(x_j)}\,\bar{g}_2(x_j)h.$$

Then A^n is a compact operator over this Hilbert space, so that the formulas for operational calculus apply. Thus,

$$\begin{aligned}\phi^n(x,t) &= (1+\tau A^n)^k\phi^n(x,0) = (1+\tau A^n)^k f(x)\\ &= \frac{1}{2\pi i}\oint (1+\lambda\tau)^k R^n_\lambda f(x)\,d\lambda\end{aligned}$$

(cf. [14], p. 287).

The contour in the above integral is around the spectrum of A^n and R^n_λ is the resolvent operator of A^n for each λ.

The final result of the next two lemmas is that the spectrum of A^n lies on the negative real axis.

<u>Lemma 1</u> A^n is self adjoint.

<u>Proof:</u> Consider the "summation by parts" formula,

$$\sum_{k=0}^{n-1} p_k \Delta q_k = p_n q_n - p_0 q_0 - \sum_{k=0}^{n-1} q_{k+1} \Delta p_k .$$

Iteration yields

$$\sum_{k=1}^{n-1} p_k \Delta^2 q_{k-1} = p_n \Delta q_{n-1} - p_1 \Delta q_0 - q_n \Delta p_{n-1} + q_1 \Delta p_0 + \sum_{k=1}^{n-1} q_k \Delta^2 p_{k-1} .$$

So

$$\begin{aligned}
\langle g_1, A^n g_2 \rangle &= \sum_{k=1}^{n-1} g_1(x_k) \frac{1}{a^n(x_k)} \overline{a^n(x_k) \frac{\Delta^2}{h^2} g_2(x_{k-1}) h} \\
&= \sum_{k=1}^{n-1} g_1(x_k) \frac{\Delta^2}{h^2} \overline{g_2(x_{k-1})}\ h \\
&= \sum_{k=1}^{n-1} \overline{g_2(x_k)} \frac{\Delta^2}{h^2} g_1(x_{k-1}) h \\
&= \sum_{k=1}^{n-1} \left[a^n(x_k) \frac{\Delta^2}{h^2} g_1(x_{k-1}) \right] \frac{1}{a^n(x_k)} \overline{g_2(x_k)}\ h \\
&= \langle A^n g_1, g_2 \rangle
\end{aligned}$$

i.e. A^n is self adjoint. □

<u>Lemma 2</u> Each eigenvalue lies on the negative real axis.

<u>Proof:</u> Since A^n is self adjoint observe that $\lambda_j \|Y_j\| = \langle Y_j, A^n Y_j \rangle$ for an eigenfunction Y_j and its corresponding eigenvalue λ_j.

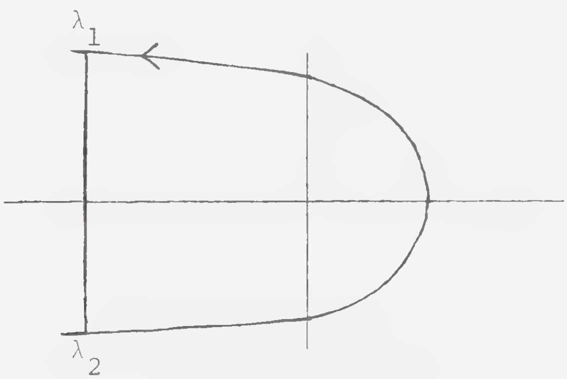
Use of the "summation by parts" formula yields

$$\langle Y_j, A^n Y_j \rangle = \sum_{k=1}^{n-1} Y_j(x_k) \frac{\Delta^2}{h^2} \overline{Y_j(x_{k-1})}\, h$$

$$= - \sum_{k=1}^{n-1} \frac{\Delta}{h} \overline{Y_j(x_k)}\, \frac{\Delta}{h} Y_j(x_k)\, h - Y_j(x_1)\overline{Y_j(x_1)}\, h \le 0 . \quad \square$$

Since the eigenvalues of A^n lie on the negative real axis, the contour of integration can be chosen as described below.

Let part of the contour coincide with the contour chosen in the continuous case, ie. $\lambda = (1-r^2) - i2r$, $-\infty < r < \infty$. Two complex numbers λ_1 and λ_2 will be found in Chapter Three which lie to the left of all the eigenvalues of A^n, and then joined to close the contour around the spectrum, as sketched below.



Summary

$$\phi(x,t) = \int_0^1 \chi_{[0,z]}(y)P_t(x,dy) - (1-x)$$

$$\phi^n(x,t) = \int_0^1 \chi_{[0,z]}(y)P_t^n(x,dy) - (1-x)$$

$$\phi(x,t) = \frac{1}{2\pi i}\oint e^{\lambda t}R_\lambda f(x)\,d\lambda$$

$$\lambda = (1-r^2) - i2r, \quad -\infty < r < \infty$$

$$\phi^n(x,t) = \frac{1}{2\pi i}\oint (1+\lambda\tau)^k R_\lambda^n f(x)\,d\lambda$$

$$R_\lambda = (\lambda - A)^{-1}$$

$$R_\lambda^n = (\lambda - A^n)^{-1}$$

$$Ag(x) = a^n(x)\,\frac{d^2}{dx^2}\,g(x)$$

$$A^n g(x) = a^n(x)\,\frac{\Delta^2}{h^2}\,g(x-h)$$

$$t = k\tau, \quad \tau = \frac{1}{4\|a^n\|_\infty n^2}$$

$$f(x) = \begin{cases} x & \text{if } x \leq z \\ x - 1 & \text{if } z < x \end{cases}$$

CHAPTER THREE

It is the purpose of this chapter to show that

$$|\phi(x,t) - \phi^n(x,t)| < K/n \quad \text{where,}$$

$$\phi(x,t) = \frac{1}{2\pi i} \oint e^{\lambda t} R_\lambda f(x)\, d\lambda$$

and

$$\phi^n(x,t) = \frac{1}{2\pi i} \oint (1+\lambda\tau)^k R_\lambda^n f(x)\, d\lambda.$$

It was shown in Chapter Two that this is equivalent to the main result, pending the exact description of the closed contour of the second integral. The assumption $z \in [0,1)$ is still kept. Also, the restriction and definitions on page 11 still hold.

To achieve the above inequality, the contours of both integrals will be truncated. Then the integrands will be shown to be small on the left portions of the contours, and close to each other on the remainder.

The appropriate bounds will be developed in sections one through five. In section one R_λ and R_λ^n will be examined in terms of Green's functions in a general setting. In section two details of the Green's functions are developed for this specific application. Section three has calculations which give a bound for the left most eigenvalue of A^n, and hence the description of the closed contour can be completed.

Section four contains a list of technical lemmas and their proofs, which are later used for the comparison of terms. Section five finally has the proof of the inequality which begins this chapter.

Section One

Observe that R_λ has the following representation [see appendix]

$$R_\lambda f(x) = \int_0^1 G_\lambda(x,y)f(y)dy.$$

In the above expression

$$G_\lambda(x,y) = \frac{1}{a^n(y)w(\lambda)} \begin{cases} Y_1(\lambda,y)Y_0(\lambda,x), & x \leq y \\ Y_0(\lambda,y)Y_1(\lambda,x), & y \leq x, \end{cases}$$

where $Y_0(\lambda,x)$ is the solution to

$$a^n(x)\frac{d^2}{dx^2}Y_0(\lambda,x) = \lambda Y_0(\lambda,x), \quad x \in (0,1),$$

$$Y_0(\lambda,0) = 0, \quad \frac{d}{dx}Y_0(\lambda,0) = 1$$

and $Y_1(\lambda,x)$ is the solution to

$$a^n(x) \frac{d^2}{dx^2} Y_1(\lambda,x) = \lambda Y_1(\lambda,x), \quad x \in (0,1),$$

$$Y_1(\lambda,1) = 0, \quad \frac{d}{dx} Y_1(\lambda,1) = -1.$$

Also,

$$w(\lambda) = Y_0(\lambda,1).$$

For a parallel expression for R^n_λ first define $Y^n_0(\lambda,x)$ and $Y^n_1(\lambda,x)$ as follows.

Let $Y^n_0(\lambda,x)$ be the solution to

$$a^n(x) \frac{\Delta^2_x}{h^2} Y^n_o(\lambda,x-h) = \lambda Y^n_0(\lambda,x), \quad x = jh, \quad 0 < j < n,$$
$$h = 1/n, \quad j \text{ an integer},$$

$$Y^n_0(\lambda,0) = 0, \quad \frac{\Delta_x}{h} Y_0(\lambda,0) = 1.$$

Let $Y^n_1(\lambda,x)$ be the solution to

$$a^n(x) \frac{\Delta^2_x}{h^2} Y^n_1(\lambda,x-h) = \lambda Y^n_1(\lambda,x), \quad x = jh, \quad 0 < j < n,$$

$$Y^n_1(\lambda,1) = 0, \quad \frac{\Delta_x}{h} Y_1(\lambda,1-h) = -1.$$

Then let,

$$w^n(\lambda) = -\left[Y_0^n(\lambda,x)\ \frac{\Delta_x}{h}\ Y_1^n(\lambda,x) - Y_1^n(\lambda,x)\ \frac{\Delta_x}{h}\ Y_0^n(\lambda,x)\right],$$

for $x = jh$, $0 \leq j < n$, j an integer.

To show that $w^n(\lambda)$ does not depend on x, it will be shown that $\frac{\Delta_x}{h} w^n(\lambda) = 0$. First notice this product formula for differences,

$$\Delta[f(x)g(x)] = g(x)\Delta f(x) + f(x+h)\Delta g(x) \ .$$

Then

$$\frac{\Delta_x}{h} w^n(\lambda) = -\left[\frac{\Delta_x}{h}\ Y_0^n(\lambda,x)\ \frac{\Delta_x}{h}\ Y_1^n(\lambda,x) + Y_0^n(\lambda,x+h)\frac{\Delta_x^2}{h^2}\ Y_1^n(\lambda,x)\right.$$

$$\left. - \frac{\Delta_x}{h}\ Y_1^n(\lambda,x)\ \frac{\Delta_x}{h}\ Y_0^n(\lambda,x) - Y_1^n(\lambda,x+h)\ \frac{\Delta_x^2}{h^2}\ Y_0^n(\lambda,x)\right]$$

$$= -\left[Y_0^n(\lambda,x+h)\ \frac{\lambda}{a^n(x+h)}\ Y_1^n(\lambda,x+h)\right.$$

$$\left. -Y_1^n(\lambda,x+h)\ \frac{\lambda}{a^n(x+h)}\ Y_0^n(\lambda,x+h)\right]$$

$$= 0.$$

Also if $x = 1 - h$ is used to calculate $w^n(\lambda)$ it can be seen that $w^n(\lambda) = Y_0^n(\lambda,1)$.

Now let

$$G_\lambda^n(x,y) = \frac{1}{a^n(y)w^n(\lambda)} \begin{cases} Y_1^n(\lambda,y)Y_0^n(\lambda,x), & x \leq y \\ Y_0^n(\lambda,y)Y_1^n(\lambda,x), & y \leq x. \end{cases}$$

Observe that

$$a^n(x)\,\frac{\Delta_x^2}{h^2}\,G_\lambda^n(x-h,y) = \lambda G_\lambda^n(x,y)$$

for $x \neq y$, $x = jh$, $0 < j < n$, j an integer.

The next series of calculations will show that

$$a^n(u)\,\frac{\Delta_x^2}{h^2}\sum_{j=1}^{n-1} G_\lambda^n(u-h,y_j)f(y_j)h = \lambda \sum_{j=1}^{n-1} G_\lambda^n(u,y_j)f(y_j)h - f(u),$$

where $y_j = jh$. This will prove that

$$R_\lambda^n f(u) = \sum_{j=1}^{n} G_\lambda^n(u,y_j)f(y_j)h.$$

The calculation begins by considering

$$a^n(u)\,\frac{\Delta_x^2}{h^2}\sum_{j=1}^{n-1} G_\lambda^n(u-h,y_j)f(y_j)h =$$

$$= \sum_{y_j \neq u} \lambda G_\lambda^n(u,y_j) f(y_j) h + f(u)\, a^n(u) \frac{\Delta_x^2}{h} G_\lambda^n(u-h,u)$$

$$= \sum_{y_j \neq u} \lambda G_\lambda^n(u,y_j) f(y_j) h + \frac{f(u) a^n(u)}{a^n(u) w^n(\lambda)} \Big[Y_0^n(\lambda,u) \frac{\Delta_x}{h} Y_1^n(\lambda,u)$$

$$- Y_1^n(\lambda,u) \frac{\Delta_x}{h} Y_0^n(\lambda,u-h) \Big]$$

$$= \sum_{y_j \neq u} \lambda G_\lambda^n(u,y_j) f(y_j) h + a^n(u) \frac{1}{h} \left[\frac{Y_1^n(\lambda,u) \Delta_x Y_0^n(\lambda,u)}{a^n(u) w^n(\lambda)} \right.$$

$$\left. - \frac{Y_1^n(\lambda,u) \Delta_x Y_0^n(\lambda,u-h)}{a^n(y) w^n(\lambda)} \right] f(u) h$$

$$+ \frac{f(u)}{w^n(\lambda)} \left[Y_0^n(\lambda,u) \frac{\Delta_x}{h} Y_1^n(\lambda,u) - Y_1^n(\lambda,u) \frac{\Delta_x}{h} Y_0^n(\lambda,u) \right]$$

$$= \sum_{y_j \neq u} \lambda G_\lambda^n(u,y_j) f(y_j) h + \lambda G_\lambda^n(u,u) f(u) h + \frac{f(u)}{w^n(\lambda)} (-w^n(\lambda))$$

$$= \lambda \sum_{j=1}^{n} G_\lambda(u,y_j) f(y_j) h - f(u).$$

Section Two

In this section further information on $Y_0(\lambda,x)$ and $Y_1(\lambda,x)$ can be obtained from the restriction that $a^n(x)$ is piecewise constant. The expression of $Y_0(\lambda,x)$ on the sub-interval $[0,x_1]$ can be used to find the expression for $Y_0(\lambda,x)$ on the sub-interval $[x_1,x_2]$. In fact the expression of $Y_0(\lambda,x)$ can be found on any sub-interval by finite recursion from the left. A similar recurrence relation will be

found for $Y_0^n(\lambda,x)$. Comparison of these recurrence relations will give a comparison of $Y_0(\lambda,x)$ and $Y_0^n(\lambda,x)$ which in turn aids the comparison of R_λ and R_λ^n.

This special comparison of solutions to difference and differential equations allows the consideration of the assumption that $\operatorname{Im}\sqrt{-\lambda}$ is 1. Otherwise the difference between $Y_0^n(\lambda,x)$ and $Y_0(\lambda,x)$ may be of order $e^{|\lambda|}$, which is not good enough.

To begin the analysis of $Y_0(\lambda,x)$, note that since $a^n(x)$ is constant on each half open interval $(x_{j-1},x_j]$, there are constants α_j and β_j, so that

$$Y_0(\lambda,x) = \alpha_j \sin\sqrt{\frac{-\lambda}{a_j}}\,x + \beta_j \cos\sqrt{\frac{-\lambda}{a_j}}\,x$$

$$\text{for } x_{j-1} < x \le x_j \text{ and } a_j = a^n(x_j).$$

Also

$$\frac{d}{dx} Y_0(\lambda,x) = \alpha_j \sqrt{\frac{-\lambda}{a_j}} \cos\sqrt{\frac{-\lambda}{a_j}}\,x - \beta_j \sqrt{\frac{-\lambda}{a_j}} \sin\sqrt{\frac{-\lambda}{a_j}}\,x$$

$$\text{for } x_{j-1} < x < x_j.$$

Next Cramer's rule will be used to solve for $\begin{pmatrix}\alpha_j\\ \beta_j\end{pmatrix}$.

Let

$$D = \sqrt{\frac{-\lambda}{a_j}}\left(-\sin^2\sqrt{\frac{-\lambda}{a_j}}\,x - \cos^2\sqrt{\frac{-\lambda}{a_j}}\,x\right)$$

$$= -\sqrt{\frac{-\lambda}{a_j}}$$

Then

$$\alpha_j = \left(-\sqrt{\frac{-\lambda}{a_j}}\, Y_0(\lambda,x) \sin\sqrt{\frac{-\lambda}{a_j}}\, x - \frac{d}{dx} Y_0(\lambda,x) \cos\sqrt{\frac{-\lambda}{a_j}}\, x\right)/D ,$$

$$\beta_j = \left(\frac{d}{dx} Y_0(\lambda,x) \sin\sqrt{\frac{-\lambda}{a_j}}\, x - \sqrt{\frac{-\lambda}{a_j}}\, Y_0(\lambda,x) \cos\sqrt{\frac{-\lambda}{a_j}}\, x\right)/D.$$

After simplifying

$$\alpha_j = Y_0(\lambda,x) \sin\sqrt{\frac{-\lambda}{a_j}}\, x + \sqrt{\frac{a_j}{-\lambda}}\, \frac{d}{dx} Y_0(\lambda,x) \cos\sqrt{\frac{-\lambda}{a_j}}\, x,$$

$$\beta_j = Y_0(\lambda,x) \cos\sqrt{\frac{-\lambda}{a_j}}\, x - \sqrt{\frac{a_j}{-\lambda}}\, \frac{d}{dx} Y_0(\lambda,x) \sin\sqrt{\frac{-\lambda}{a_j}}\, x,$$

$$\text{for} \quad x_{j-1} < x < x_j.$$

Also, since $Y_0(\lambda,x)$ is continuously differentiable,

$$Y_0(\lambda,x_j^-) = Y_0(\lambda,x_j^+) \quad \text{and} \quad \frac{d}{dx} Y_0(\lambda,x_j^-) = \frac{d}{dx} Y_0(\lambda,x_j^+),$$

$$\text{for} \quad 0 < j < n.$$

Thus,

$$\alpha_{j+1} = Y_0(\lambda,x_j^-) \sin\sqrt{\frac{-\lambda}{a_{j+1}}}\, x_j + \sqrt{\frac{a_{j+1}}{-\lambda}}\, \frac{d}{dx} Y_0(\lambda,x_j^-) \cos\sqrt{\frac{-\lambda}{a_{j+1}}}\, x_j ,$$

$$\beta_{j+1} = Y_0(\lambda,x_j^-) \cos \sqrt{\frac{-\lambda}{a_{j+1}}}\, x_j - \sqrt{\frac{a_{j+1}}{-\lambda}}\, \frac{d}{dx} Y_0(\lambda,x_j^-) \sin \sqrt{\frac{-\lambda}{a_{j+1}}}\, x_j .$$

Next, the appropriate substitution for $Y_0(\lambda,x_j^-)$ and $\frac{d}{dx} Y_0(\lambda,x_j^-)$ gives

$$\alpha_{j+1} = \left(\alpha_j \sin \sqrt{\frac{-\lambda}{a_j}}\, x_j + \beta_j \cos \sqrt{\frac{-\lambda}{a_j}}\, x_j\right) \sin \sqrt{\frac{-\lambda}{a_{j+1}}}\, x_j$$

$$+ \sqrt{\frac{a_{j+1}}{a_j}} \left(\alpha_j \cos \sqrt{\frac{-\lambda}{a_j}}\, x_j - \beta_j \sin \sqrt{\frac{-\lambda}{a_j}}\, x_j\right) \cos \sqrt{\frac{-\lambda}{a_{j+1}}}\, x_j ,$$

$$\beta_{j+1} = \left(\alpha_j \sin \sqrt{\frac{-\lambda}{a_j}}\, x_j + \beta_j \cos \sqrt{\frac{-\lambda}{a_j}}\, x_j\right) \cos \sqrt{\frac{-\lambda}{a_{j+1}}}\, x_j$$

$$- \sqrt{\frac{a_{j+1}}{a_j}} \left(\alpha_j \cos \sqrt{\frac{-\lambda}{a_j}}\, x_j - \beta_j \sin \sqrt{\frac{-\lambda}{a_j}}\, x_j\right) \sin \sqrt{\frac{-\lambda}{a_{j+1}}}\, x_j .$$

A rearrangement of terms gives

$$\alpha_{j+1} = \alpha_j \cos\left(\sqrt{-\lambda}\, x_j \left[\frac{1}{\sqrt{a_{j+1}}} - \frac{1}{\sqrt{a_j}}\right]\right) + \beta_j \sin\left(\sqrt{-\lambda}\, x_j \left[\frac{1}{\sqrt{a_{j+1}}} - \frac{1}{\sqrt{a_j}}\right]\right)$$

$$+ \left(\sqrt{\frac{a_{j+1}}{a_j}} - 1\right)\left(\alpha_j \cos \sqrt{\frac{-\lambda}{a_j}}\, x_j - \beta_j \sin \sqrt{\frac{-\lambda}{a_j}}\, x_j\right) \cos \sqrt{\frac{-\lambda}{a_{j+1}}}\, x_j ,$$

$$\beta_{j+1} = \alpha_j \sin\left(\sqrt{-\lambda}\, x_j \left[\frac{1}{\sqrt{a_j}} - \frac{1}{\sqrt{a_{j+1}}}\right]\right) + \beta_j \cos\left(\sqrt{-\lambda}\, x_j \left[\frac{1}{\sqrt{a_j}} - \frac{1}{\sqrt{a_{j+1}}}\right]\right)$$

$$+\left(1-\sqrt{\frac{a_{j+1}}{a_j}}\right)\left(\alpha_j \cos\sqrt{\frac{-\lambda}{a_j}}\, x_j - \beta_j \sin\sqrt{\frac{-\lambda}{a_j}}\, x_j\right)\sin\sqrt{\frac{-\lambda}{a_{j+1}}}\, x_j .$$

The last two equations give a relation between $\begin{pmatrix}\alpha_j\\ \beta_j\end{pmatrix}$ and $\begin{pmatrix}\alpha_{j+1}\\ \beta_{j+1}\end{pmatrix}$.

Also $\frac{d}{dx} Y_0(\lambda,0) = 1$, so

$$\alpha_1 = \sqrt{\frac{a_1}{-\lambda}} \quad \text{and} \quad \beta_1 = 0.$$

Thus $Y_0(\lambda,x)$ is completely defined on $[0,1]$ by the recurrence relation and initial values.

Properties of $Y_1(\lambda,x)$ can be deduced from $Y_0(\lambda,x)$ because of their symmetric relationship.

In a similar manner $Y_0^n(\lambda,x)$ will be analyzed. First recall some trigonometry

$$\sin(a+b) - \sin(a-b) = 2 \sin b \cos a,$$

$$\cos(a+b) - \cos(a-b) = -2 \sin b \sin a.$$

Thus,

$$\Delta \sin(ux) = \sin(u(x+h)) - \sin(ux)$$

$$= 2 \sin \frac{uh}{2} \cos\left(u(x+\frac{h}{2})\right)$$

and

$$\Delta \cos(ux) = \cos(u(x+h)) - \cos(ux)$$

$$= -2 \sin \frac{uh}{2} \sin\left(u(x+\frac{h}{2})\right) .$$

Iteration gives,

$$\Delta^2 \sin(u(x-h)) = -4 \sin^2 \frac{uh}{2} \sin(ux),$$

$$\Delta^2 \cos(u(x-h)) = -4 \sin^2 \frac{uh}{2} \cos(ux).$$

Keeping these relations in mind, artificially define $Y_0^n(\lambda,x)$ point-wise by

$$Y_0^n(\lambda,x) = \alpha_j^n \sin u_j x + \beta_j^n \cos u_j x,$$

$$\text{for} \quad x = x_j \ , \quad j = 1,2,\ldots,n \ .$$

The parameters α_j^n, u_j and β_j^n will be chosen so that $Y_0^n(\lambda,x)$ satisfies its defining difference equation and initial conditions.

Since, $Y_0^n(\lambda,x)$ must satisfy the second order difference equation, let

$$u_j = \frac{2}{h} \sin^{-1} \frac{h}{2} \sqrt{\frac{-\lambda}{a_j}} \ .$$

Then

$$Y_o^n(\lambda,x) = \alpha_j^n \sin u_j x + \beta_j^n \cos u_j x$$

for $x = x_{j-1}, x_j, x_{j+1}$.

Also,

$$\frac{\Delta_x}{h} Y_0^n(\lambda,x) = \alpha_j^n \sqrt{\frac{-\lambda}{a_j}} \cos u_j(x+\frac{h}{2}) - \beta_j^n \sqrt{\frac{-\lambda}{a_j}} \sin u_j(x+\frac{h}{2})$$

for $x = x_{j-1}$ and $x = x_j$.

These two linear equations will be solved for $\begin{pmatrix} \alpha_j^n \\ \beta_j^n \end{pmatrix}$.

To solve the linear equations use Cramer's rule. Let,

$$D = \sqrt{\frac{-\lambda}{a_j}} \left[-\sin u_j x \sin u_j(x+\frac{h}{2}) - \cos u_j x \cos u_j(x+\frac{h}{2}) \right]$$

$$= - \sqrt{\frac{-\lambda}{a_j}} \cos u_j \frac{h}{2} = - \sqrt{\frac{-\lambda}{a_j}} \sqrt{1+ \frac{h^2\lambda}{4a_j}} \quad .$$

Then,

$$\alpha_j^n = \left[- \sqrt{\frac{-\lambda}{a_j}} Y_0^n(\lambda,x) \sin u_j(x+\frac{h}{2}) - \frac{\Delta_x}{h} Y_0^n(\lambda,x) \cos u_j x \right] / D$$

$$\beta_j^n = \left[\frac{\Delta_x}{h} Y_0^n(\lambda,x)\sin u_j(x+\frac{h}{2}) - \sqrt{\frac{-\lambda}{a_j}}\, Y_0^n(\lambda,x)\cos u_j(x+\frac{h}{2})\right]/D.$$

Simplification gives

$$\alpha_j^n = \left[Y_0^n(\lambda,x)\sin u_j(x+\frac{h}{2}) + \sqrt{\frac{a_j}{-\lambda}}\,\frac{\Delta_x}{h}\, Y_0^n(\lambda,x)\cos u_j x\right]\Big/\sqrt{1+\frac{h^2\lambda}{4a_j}} \quad ,$$

$$\beta_j^n = \left[Y_0^n(\lambda,x)\cos u_j(x+\frac{h}{2}) - \sqrt{\frac{a_j}{-\lambda}}\,\frac{\Delta_x}{h}\, Y_0^n(\lambda,x)\sin u_j x\right]\Big/\sqrt{1+\frac{h^2\lambda}{4a_j}} \quad .$$

The above expressions are true for $x = x_{j-1}$, $x = x_j$, $1 + \frac{h^2\lambda}{4a_j} \neq 0$, and $1 \leq j \leq n-1$.

So, for $1 \leq j \leq n-1$,

$$\alpha_{j+1}^n = \left[Y_0^n(\lambda,x_j)\sin u_{j+1}(x_j+\frac{h}{2}) + \sqrt{\frac{a_{j+1}}{-\lambda}}\,\frac{\Delta_x}{h}\, Y_0^n(\lambda,x_j)\cos u_{j+1}x_j\right]\Big/\sqrt{1+\frac{h^2\lambda}{4a_{j+1}}}$$

$$\beta_{j+1}^n = \left[Y_0^n(\lambda,x_j)\cos u_{j+1}(x_j+\frac{h}{2}) - \sqrt{\frac{a_{j+1}}{-\lambda}}\,\frac{\Delta_x}{h}\, Y_0^n(\lambda,x_j)\sin u_{j+1}x_j\right]\Big/\sqrt{1+\frac{h^2\lambda}{4a_{j+1}}} \quad .$$

Substitution for Y_0^n and $\frac{\Delta_x}{h} Y_0^n$ gives

$$\alpha_{j+1}^n = \Bigg[(\alpha_j^n \sin u_j x_j + \beta_j^n \cos u_j x_j)\sin u_{j+1}(x_j+\frac{h}{2})$$

$$+ \sqrt{\frac{a_{j+1}}{a_j}}\left(\alpha_j^n \cos u_j(x_j+\frac{h}{2}) - \beta_j^n \sin u_j(x_j+\frac{h}{2})\right)\cos u_{j+1}x_j\Bigg]\Big/\sqrt{1+\frac{h^2\lambda}{4a_{j+1}}} \quad ,$$

$$\beta^n_{j+1} = \Bigg[(\alpha^n_j \sin u_j x_j + \beta^n_j \cos u_j x_j) \cos u_{j+1}(x_j + \tfrac{h}{2})$$

$$- \sqrt{\frac{a_{j+1}}{a_j}} \left(\alpha^n_j \cos u_j (x_j + \tfrac{h}{2}) - \beta_j \sin u_j (x_j + \tfrac{h}{2}) \right) \sin u_{j+1} x_j \Bigg] \Big/ \sqrt{1 + \frac{h^2\lambda}{4a_{j+1}}} .$$

The last two equations give a relation between $\begin{pmatrix} \alpha^n_j \\ \beta^n_j \end{pmatrix}$ and $\begin{pmatrix} \alpha^n_{j+1} \\ \beta^n_{j+1} \end{pmatrix}$. The condition that $\frac{\Delta_x}{h} Y^n_0(\lambda, 0) = 1$, puts a restriction on $\begin{pmatrix} \alpha^n_1 \\ \beta^n_1 \end{pmatrix}$

$$\alpha^n_1 = \sqrt{\frac{a_1}{-\lambda}} \Big/ \cos(u_1 \tfrac{h}{2}) = \sqrt{\frac{a_1}{-\lambda}} \Big/ \sqrt{1 + \frac{h^2\lambda}{4a_1}}$$

$$\beta^n_1 = 0$$

The recurrence relation and initial condition thus uniquely determines $\begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix}$ and $Y^n_0(\lambda, x_j)$, for $1 \leq j \leq n$.

This construction is valid for λ on the contour $\sqrt{-\lambda} = r + i$ since $1 + \frac{h^2\lambda}{4a_j} \neq 0$ for any j and $\frac{h}{2}\sqrt{\frac{-\lambda}{a_j}}$ is in the domain of $\sin^{-1}$.

Many properties of $Y^n_1(\lambda, x)$ may be deduced from $Y^n_0(\lambda, x)$ because of their symmetric relationship.

Section Three

To complete the description of the contour around the finite spectrum of A^n, a bound for the left most eigenvalue of A^n must be found.

Note that an eigenvalue λ^n_k has an eigenfunction $Y^n_0(\lambda^n_k, x)$ associated with it which solves

1) $a_j \dfrac{\Delta_x^2}{h^2} Y_0^n(\lambda_k^n, x_{j-1}) = \lambda_k^n Y_0^n(\lambda_k^n, x_j)$ for $0 < j < n$,

2) $Y_0^n(\lambda_k^n, x_1) = h$,

3a) $Y_0^n(\lambda_k^n, x_0) = 0$,

3b) $Y_0^n(\lambda_k^n, x_n) = 0$.

It will be shown that if $\lambda \le -4\|a^n\|_\infty n^2$ and if $Y_0^n(\lambda, x)$ satisfies 1), 2), and 3a) then

$$|Y_0^n(\lambda, x_n)| \ge h$$

so that $Y_0^n(\lambda, x)$ does not satisfy 3b), and λ is not an eigenvalue.

Temporarily define

$$b_j = Y_0^n(\lambda, x_j)$$

and

$$M_j = -2 - \frac{h^2\lambda}{a_j} .$$

The difference equation for $Y_0^n(\lambda, x_j)$ becomes $b_{j+1} = -M_j b_j - b_{j-1}$. If $\lambda \le -4\|a^n\|_\infty \dfrac{1}{h^2}$ then $2 \le |M_j|$. Also

$$\Big| \, |M_j||b_j| - |b_{j-1}| \, \Big| \le |b_{j+1}| .$$

An induction argument then shows $h \le |b_j| \le |b_{j+1}|$.

□

Section Four

The statements of thirteen technical lemmas will be listed in this section. Then their proofs will follow. First four points on the infinite contour will be defined.

Let $$\lambda_1 = -(1+\frac{1}{\tau}) + i\ 2\sqrt{1+(1+ 1/\tau)},$$

$$\lambda_2 = -(1+\frac{1}{\tau}) - i\ 2\sqrt{1+(1+ 1/\tau)}.$$

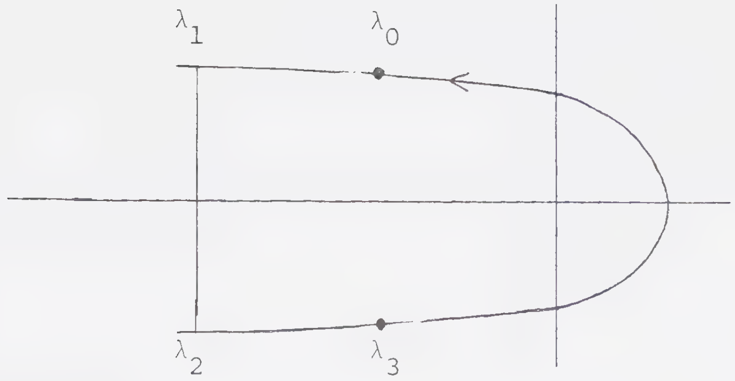
since $-1/\tau = -4||a^n||_\infty \frac{1}{h^2}$, λ_1 and λ_2 are left of all the eigenvalues of A^n. The closed contour around the eigenvalues of A^n, is formed from the open contour by connecting λ_1 to λ_2, by a vertical line.

Truncation points λ_0 and λ_3 of both contours are defined below.

Let $$\check{a} = \min\{a^n(x_j): 0 < j < n\}$$

Let $$\lambda_0 = -2\check{a}n^{1/3} + i\ 2\sqrt{1+2\check{a}n^{1/3}}$$

and $$\lambda_3 = -2\check{a}n^{1/3} - i\ 2\sqrt{1+2\check{a}n^{1/3}}.$$



Lemmas to be used later are listed as follows.

<u>Lemma 1</u> Let $Y(x)$ be the complex valued solution to

$$g(x)Y''(x) = \lambda Y(x),$$

$$Y(0) = 0, \quad Y'(0) = 1, \quad x \in [0,1].$$

Let $g \in C^2(0,1)$, $g(x) > \varepsilon > 0$ for $x \in (0,1)$, and $g(x)$ bounded for $x \in [0,1]$. Let $\sqrt{-\lambda} = r + i$, $-\infty < r < \infty$.

Then

i) $\sqrt{|\lambda|}\, Y(x)$ is bounded for $x \in [0,1]$.

ii) $Y'(x)$ is bounded for $x \in (0,1)$.

<u>Lemma 2</u> The following functions are uniformly bounded in n and λ, for $x \in [0,1]$, $\sqrt{-\lambda} = r + i$, $-\infty < r < \infty$.

i) $\sqrt{|\lambda|}\, Y_0(\lambda,x)$,

ii) $\sqrt{|\lambda|}\, Y_1(\lambda,x)$,

iii) $\frac{d}{dx} Y_0(\lambda,x)$,

iv) $\frac{d}{dx} Y_1(\lambda,x)$.

Lemma 3 $|w(\lambda)| > \varepsilon/\sqrt{|\lambda|}^{\,3}$ for some $\varepsilon > 0$, $\lambda = (1-r^2) - i\, 2r$, $-\infty < r < \infty$ and all $n > 0$.

Lemma 4 $|R_\lambda f(x)| \leq K_1$, for all $n > 0$, some $K_1 > 0$, and $\lambda = (1-r^2) - i\, 2r$, $-\infty < r < \infty$.

Lemma 5 $|\oint e^{\lambda t} R_\lambda f(x)\, d\lambda| \leq K_2/n$, for some constant $K_2 > 0$, and all $n > 0$, with λ on the disconnected contour $\lambda = (1-r^2) - i\, 2r$, $-\infty < r < \infty$, Re $\lambda \leq -2\check{a}n^{1/3}$.

Lemma 6 $|R_\lambda^n f(x)| \leq K_3\sqrt{n}$ for some $K_3 > 0$ and all $n > 0$, with $\lambda = (1-r^2) - i\, 2r$, $-\infty < r < \infty$, Re $\lambda \leq -2\check{a}n^{1/3}$ or Re $\lambda = -\frac{1}{\tau} - 1$ and $|\text{Im } \lambda| \leq 2\sqrt{2+ 1/\tau}$.

Lemma 7 $|\oint (1+\lambda\tau)^k R_\lambda^n f(x)\, d\lambda| < K_4/n$ for some $K_4 > 0$ and all n large enough, with $\lambda = (1-r^2) - i\, 2r$, $-\frac{1}{\tau} - 1 \leq$ Re $\lambda \leq -2\check{a}n^{1/3}$ or Re $\lambda = -\frac{1}{\tau} - 1$ with $|\text{Im } \lambda| \leq 2\sqrt{1+2\check{a}n^{1/3}}$.

Lemma 8 Let p_j be a solution to the difference equation

$$\frac{\Delta}{h} p_j = b_j + c_j p_j .$$

Then,

$$p_j = \left[p_1 + \sum_{i=2}^{j} b_{i-1} \prod_{k=2}^{i} (1+hc_{k-1})h \right] \prod_{i=2}^{j} (1+hc_{j-1}) \ .$$

Lemma 9 Consider the recursion formulas

$$p(x_{j+1}) = F\left(x_j, p(x_j)\right), \qquad q(x_{j+1}) = G\left(x_j, q(x_j)\right).$$

Let $\Delta x_j = h$ for all j. Suppose G is continuously differentiable in its second variable, with F, G and $\frac{\partial}{\partial q} G$ bounded with respect to both variables. Then

$$|p(x_j)-q(x_j)| \leq \left(|p(x_1)-q(x_1)| + m^{j-1}(j-1)||F-G||_\infty \right) ||\frac{\partial}{\partial q} G||_\infty^{j-1} \ .$$

In the above expression, the derivative is taken with respect to the second variable. The sup norms are taken with respect to both independent variables. Also m is defined by

$$m = \max \left\{ || (\frac{\partial}{\partial q} G)^{-1} ||_\infty , 1 \right\}.$$

Lemma 10 Consider the vector recurrence formulas

$$\begin{pmatrix} \alpha(x_{j+1}) \\ \beta(x_{j+1}) \end{pmatrix} = F\left[x_j, \begin{pmatrix} \alpha(x_j) \\ \beta(x_j) \end{pmatrix} \right],$$

$$\begin{pmatrix} \bar{\alpha}(x_{j+1}) \\ \bar{\beta}(x_{j+1}) \end{pmatrix} = G\left[x_j, \begin{pmatrix} \bar{\alpha}(x_j) \\ \bar{\beta}(x_j) \end{pmatrix}\right].$$

Let $\Delta x_j = h$ for all j.

Suppose G has a continuous Jacobian in its second variable, with F, G and Jacobian of G bounded with respect to both variables. Then,

$$\left|\begin{pmatrix} \alpha(x_j) \\ \beta(x_j) \end{pmatrix} - \begin{pmatrix} \bar{\alpha}(x_j) \\ \bar{\beta}(x_j) \end{pmatrix}\right| = \left(\left|\begin{pmatrix} \alpha(x_1) \\ \beta(x_1) \end{pmatrix} - \begin{pmatrix} \bar{\alpha}(x_1) \\ \bar{\beta}(x_1) \end{pmatrix}\right| + m^{j-1}(j-1)\|F-G\|_\infty\right)\|JG\|_\infty^{j-1}.$$

In the above expression JG is the Jacobian of G in its second variable. The sup norms are in both variables. m is defined by

$$m = \max\left\{\|(JG)^{-1}\|_\infty, 1\right\}.$$

<u>Lemma 11</u> The finite sequences α_j and β_j are uniformly bounded in n, j and λ for $1 \leq j \leq n$, $\lambda = (1-r^2) - i\,2r$, $-\infty < r < \infty$.

<u>Lemma 12</u> If $\lambda = (1-r^2) - i\,2r$, with $-2\overset{\vee}{a}n^{1/3} \leq \mathrm{Re}\,\lambda$ and $x_j = jh = \frac{j}{n}$, $1 \leq j \leq n$, then the following inequalities hold for some constants K_5, K_6 and n large enough

i) $|Y_0(\lambda,x_j) - Y_0^n(\lambda,x_j)| \leq K_5|\lambda|/n$

ii) $|Y_1(\lambda,x_j) - Y_1^n(\lambda,x_j)| \leq K_6|\lambda|/n$

Lemma 13

$$\left| \int_{\lambda_3}^{\lambda_0} \left(e^{\lambda t} - (1+\lambda\tau)^k \right) R_\lambda f(x)\, d\lambda \right| \leq K_6/n$$

for some constant K_6 and all n large enough.

Proof of Lemma 1: i) This proof is inspired by Liapounov or energy function techniques. Let $Y(x) = U(x) + iV(x)$ for U, V real. Then the equation

$$gY'' = Y, \quad Y(0) = 0, \quad Y'(0) = 1$$

becomes

$$\begin{cases} U'' + [(r^2-1)U - 2rV]/g = 0 \\ V'' + [2rU + (r^2-1)V]/g = 0 \end{cases}$$

$$U(0) = 0, \quad U'(0) = 1$$

$$V(0) = 0, \quad V'(0) = 0.$$

Case I. Let $r \geq 3$.

Let

$$E = U'^2 + V'^2 + [(r^2-1)(U^2+V^2) + 4rUV]/g.$$

It will be shown that, if Y is unbounded on the domain in question,

then E will be unbounded.

Next it will be shown that E is bounded which will prove 1 (i).

Note that $r^2 - 1 \geq \frac{8}{9} r^2$ and $4r \leq \frac{4}{3} r^2$ so

$$[(r^2-1)(U^2+V^2)+4rUV] \geq \frac{3}{4} (r^2-1)(U^2+V^2) + 4rUV$$

$$\geq \frac{2}{3} r^2(U^2+V^2) - \frac{4}{3} r^2|U||V|$$

$$= \frac{2}{3} r^2(|U - V|)^2$$

$$\geq 0.$$

Then

$$E \geq U'^2 + V'^2 + \frac{1}{4g} (r^2-1)(U^2+V^2)$$

$$\geq \frac{1}{4\|g\|_\infty} |\lambda| \; |Y|^2 \quad .$$

Thus if $\sqrt{|\lambda|}\, Y$ is unbounded, E is unbounded.

Calculation of $\frac{d}{dx} E$ follows.

$$\frac{d}{dx} E = E' = 2U'U'' + 2V'V''$$

$$+ [(r^2-1)(2UU'+2VV')+4r(U'V+UV')]/g$$

$$- \frac{g'}{g^2} [(r^2-1)(U^2+V^2)+4rUV].$$

Use of the original second order differential equations yield substitutions for U'' and V''. Some of the resulting terms cancel, leaving

$$E' = 8\,\frac{rU'V}{g} - \frac{g'}{g^2}\,[(r^2-1)(U^2+V^2)+4rUV].$$

Next some trickery will yield a bound for $\frac{8rU'V}{g}$. Note that

$$0 \le \left(|U'| - \frac{1}{2\sqrt{g}}\,|r|\,|V|\right)^2.$$

So

$$0 \le U'^2 - \frac{1}{\sqrt{g}}\,|r|\,|U'|\,|V| + \frac{1}{4g}\,r^2V^2.$$

This gives

$$\begin{aligned}\frac{rU'V}{\sqrt{g}} &\le U'^2 + \frac{1}{4g}\,r^2V^2\\ &\le U'^2 + \frac{1}{g}\,(r^2-1)V^2 \le 4E.\end{aligned}$$

Thus $\frac{8rU'V}{g} \le \frac{32}{\sqrt{g}}\,E$. Also no matter what the sign of g' is

$$-\frac{g'}{g^2}\,[(r^2-1)(U^2+V^2)+4rUV] \le \frac{g'}{g^2}\,E$$

Finally $E' \le \left[\frac{32}{\sqrt{g}} + \frac{g'}{g^2}\right]E$. This expression is independent of λ.

So

$$E \leq E(0)\exp\left(\int_0^1 \left|\frac{32}{\sqrt{g}}\right| dx + \int_0^1 \left|\frac{g'}{g}\right| dx\right)$$

Since $g \geq \varepsilon > 0$, the integrals are finite. Also $E(0) = 1$. So, E is bounded.

Case II. $r \leq 3$ (similar to Case I) This time let

$$E = U'^2 + V'^2 + \frac{1}{g}[(r^2+1)(U^2+V^2)+2rUV].$$

Note that

$$[(r^2+1)(U^2+V^2)+2rUV] = (rU+V)^2 + U^2 + r^2V^2 \geq 0$$

Thus if E is bounded, then $(r^2+1)(U^2+V^2)$ is bounded, to get $\sqrt{|\lambda|}\ Y$ is bounded for $x \in [0,1]$. Differentiation and substitution for U'' and V'' eventually yields.

$$E' = 4(UU'+VV')/g + 2r(3U'V-V'U)/g$$
$$+ [(r^2+1)(U^2+V^2)+2rUV]g'/g^2$$

The terms involving UU', VV', $U'V$, and $V'U$ can be bound by expressions generated from the completion of squares trick. For instance

$$0 \le \left(|U'| - \frac{1}{\sqrt{g}} |U| \right)^2$$

so

$$\frac{2}{\sqrt{g}} U|U'| \le U'^2 + \frac{U^2}{g} \le E$$

so

$$\left|\frac{4UU'}{g}\right| \le \frac{2}{\sqrt{g}} E$$

Similarly the other three terms can be bounded by expressions of the form $\frac{c}{\sqrt{g}} E$. The last term is bounded by $\frac{g'}{g^2} E$. Thus

$$E \le E(0) \exp\left(\int_0^1 \left|\frac{c}{\sqrt{g}}\right| dx + \int_0^1 \left|\frac{g'}{g}\right| dx \right) < \infty.$$

□

ii) From i) both cases

$$E \ge U'^2 + V'^2$$

Since E is bounded, U' and V' are bounded. Thus Y' is bounded.

□

Proof of Lemma 2: i) This is nearly a direct application of lemma 1 (i) except that a^n is not differentiable. However, because $a^n(x)$ is a step function approximation to $a(x)$, all the a^n's can

be bounded above and below by differentiable functions with the required properties. Then the proof of lemma 1 (i) may be mimicked, replacing the upper or lower boundary function for g in order to make the inequalities go the right way.

The same arguments apply to $Y_1(\lambda,x)$, $\frac{d}{dx}Y_0(\lambda,x)$ and $\frac{d}{dx}Y_1(\lambda,x)$ to yield ii, iii, and iv.

$\square$

Proof of Lemma 3: A lower bound for $|w(\lambda)|$ will first be found in terms of weighted L^2 norms of certain functions. Crude estimates of these norms will then be calculated.

Let $U(x) = x$.

Let $Z(\lambda,\cdot) = Y_0(\lambda,\cdot) - w(\lambda)U(\cdot)$. Note that $Z(\lambda,\cdot)$ is in the domain of A. Then,

$$(\lambda-A)Z(\lambda,\cdot) = -\lambda w(\lambda)U(\cdot).$$

Thus

$$Z(\lambda,\cdot) = -\lambda w(\lambda)R_\lambda U(\cdot),$$

and

$$\frac{Z(\lambda,\cdot)}{w(\lambda)} = -\lambda R_\lambda U(\cdot).$$

Thus,

$$\frac{1}{|w(\lambda)|} = |\lambda| \frac{\|R_\lambda U(\cdot)\|_2}{\|Z(\lambda,\cdot)\|_2} \leq$$

$$\leq |\lambda| \frac{\|R_\lambda U(\cdot)\|_2}{\left| \|Y_0(\lambda,\cdot)\|_2 - |w(\lambda)| \, \|U(\cdot)\|_2 \right|} .$$

so,

$$\left| \|Y_0(\lambda,\cdot)\|_2 - |w(\lambda)| \, \|U(\cdot)\|_2 \right| \leq |\lambda| \, \|R_\lambda U(\cdot)\|_2 \, |w(\lambda)| .$$

Case I. if $\|Y_0(\lambda,\cdot)\|_2 \leq |w(\lambda)| \, \|U(\cdot)\|_2$ then

$$\frac{\|Y_0(\lambda,\cdot)\|_2}{\|U(\cdot)\|} \leq |w(\lambda)| .$$

Case II. if $|w(\lambda)| \, \|U(\cdot)\|_2 < \|Y_0(\lambda,\cdot)\|_2$ then

$$\frac{\|Y_0(\lambda,\cdot)\|_2}{|\lambda| \, \|R_\lambda U(\cdot)\|_2 + \|U(\cdot)\|_2} \leq |w(\lambda)| .$$

The calculations of a lower bound for $\|Y_0(\lambda,\cdot)\|_2$ and upper bounds for $\|R_\lambda(U,\cdot)\|_2$ and $\|U(\cdot)\|_2$ will be done next.

The initial conditions of $Y_0(\lambda,x)$ yield the Taylor series expansion

$$Y_0(\lambda,x) = x + \frac{x^2}{2} B_1(\lambda,x) .$$

Here

$$|B_1(\lambda,x)| \leq \sup \left\{ \left| \frac{\partial^2}{\partial x^2} Y(\lambda,\xi) \right| : 0 \leq \xi \leq x \right\}$$

$$= \sup \left\{ \frac{\lambda}{2a(\xi)} \; Y(\lambda,\xi) \right| : 0 \leq \xi \leq x \right\} .$$

From Lemma 2, $|Y_0(\lambda,x)| \leq \frac{B_2}{\sqrt{|\lambda|}}$ for some $B > 0$ and $0 \leq x \leq 1$.

Temporarily let,

$$b = \frac{\sqrt{|\lambda|}\ B_2}{2 \min \{a\}} .$$

Then

$$0 \leq x - x^2 b \leq |Y_0(\lambda,x)| \qquad 0 \leq x \leq \frac{1}{b} .$$

Also

$$\|Y_0(\lambda,\cdot)\|_2^2 \geq \frac{1}{\|a\|_\infty} \int_0^{1/b} |Y_0(\lambda,x)|^2 dx$$

$$\geq \frac{1}{\|a\|_\infty} \int_0^{1/b} (x-x^2 b)^2 dx$$

$$= \frac{1}{\|a_\infty\|} \left(\frac{1}{3b^3} - \frac{2}{4b^3} + \frac{1}{5b^3} \right)$$

$$\geq \frac{c}{|\lambda|^{3/2}} \quad \text{for some } c > 0.$$

Recall $|\lambda| > 1$.

Thus $\|Y_0(\lambda,\cdot)\|_2 \geq \frac{c}{|\lambda|}$.

Next consider $\|U(\cdot)\|_2$. Since $U(\cdot)$ is bounded and all $a^n(\cdot)$ are bounded below uniformly, $\|U(\cdot)\|_2 < \infty$.

Also

$$\|R_\lambda U(\cdot)\|_2 = \|\sum_{j=1}^{\infty} \frac{1}{\lambda-\lambda_j} \langle U, Y_j\rangle Y_j\|$$

$$\leq \frac{1}{\min\{\lambda-\lambda_j\}} \|U(\cdot)\|_2 \leq \frac{\|U(\cdot)\|_2}{\sqrt{|\lambda|}}.$$

Where λ_j is the jth eigenvalue [see appendix]. The last inequality above follows from λ_j lying on the negative real axis and $\sqrt{-\lambda} = r + i$.

The expression for the lower bound of $|w(x)|$ in either case now yields

$$\frac{\varepsilon}{|\lambda|^{3/2}} \leq |w(\lambda)| \quad \text{for some} \quad \varepsilon > 0.$$

□

Proof of Lemma 4: Recall $f(x) = \chi_{[0,z]}(x) - (1-x)$ and,

$$R_\lambda f(x) = \int_0^1 G_\lambda(x,y) f(y)\,dy.$$

Since f is piecewise linear, the calculation of the above integral is simplified.

Consider for instance,

$$\int_b^c G_\lambda(x,y) f(y)\,dy$$

with $z \notin (b,c)$ and $x \leq b$.

Then

$$\int_b^c G_\lambda(x,y)f(y)dy = \frac{1}{w(\lambda)} \int_b^c \frac{1}{a(y)} Y_0(\lambda,x)Y_1(\lambda,y)f(y)dy$$

$$= \frac{Y_0(\lambda,x)}{w(\lambda)\lambda} \int_b^c \frac{d^2}{dy^2} Y_1(\lambda,y)f(y)dy$$

$$= \frac{Y_0(\lambda,x)}{w(\lambda)\lambda} \left(f(y) \frac{d}{dy} Y_1(\lambda,y)\Big|_b^c - f'(y)Y_1(\lambda,y)\Big|_b^c \right.$$

$$\left. + \int_b^c Y_1(\lambda,y)f''(y)dy \right).$$

The last integral is zero because $f''(y) = 0$.

Now suppose $x < z$.

Then

$$\int_0^1 G_\lambda(x,y)f(y)dy = \int_0^x G_\lambda(x,y)f(y)dy$$

$$+ \int_x^z G_\lambda(x,y)f(y)dy + \int_z^1 G_\lambda(x,y)f(y)dy.$$

Next observe

$$\int_0^x G_\lambda(x,y)f(y)dy = \frac{Y_1(\lambda,x)}{w(\lambda)\lambda} \left(f(x) \frac{d}{dy} Y_0(\lambda,x) - Y_0(\lambda,x) \right),$$

$$\int_x^z G_\lambda(x,y)f(y)dy = \frac{Y_0(\lambda,x)}{w(\lambda)\lambda} \left(f(z^-) \frac{d}{dy} Y_1(\lambda,z) - f(x) \frac{d}{dy} Y_1(\lambda,x) \right.$$

$$\left. - Y_1(\lambda,z) + Y_1(\lambda,x) \right),$$

$$\int_x^z G_\lambda(x,y)f(y)dy = \frac{Y_0(\lambda,x)}{w(\lambda)\lambda}\left(-f(z^+)\ \frac{d}{dy}Y_1(\lambda,z)+Y_1(\lambda,z)\right).$$

Now use the facts that

$$f(z^-) - f(z^+) = 1$$

and

$$w(\lambda) = Y_1(\lambda,x)\ \frac{d}{dy}Y_0(\lambda,x) - Y_0(\lambda,x)\ \frac{d}{dy}Y_1(\lambda,x).$$

Then

$$R_\lambda f(x) = \frac{1}{\lambda}\left(f(x)+\frac{Y_0(\lambda,x)\ \frac{d}{dy}Y_1(\lambda,z)}{w(\lambda)}\right).$$

Similarly if $z < x$

$$R_\lambda f(x) = \frac{1}{\lambda}\left(f(x)+\frac{Y_1(\lambda,x)\ \frac{d}{dy}Y_0(\lambda,z)}{w(\lambda)}\right).$$

Lemma 2 and the continuity of $R_\lambda f(x)$ now yield

$$|R_\lambda f(x)| \le \frac{B}{\sqrt{\lambda}^{\,3}}\ |w(\lambda)|,$$

for some $B > 0$.

Application of lemma 3 now yields

$$|R_\lambda f(x)| \leq \frac{B}{\varepsilon} .$$

□

Proof of Lemma 5: Consider first

$$\left|\int_{\lambda_0}^{-\infty} e^{\lambda t} R_\lambda f(x)\, d\lambda\right|$$

make the change of variables

$$s = (-\lambda)^{\frac{1}{2}}$$

$$s = r + i.$$

Then the above integral is bounded by a constant multiple of

$$\int_{-b}^{-\infty} e^{-r^2} |R_\lambda f(x)|\,|r+1|\,dr.$$

where $b = \sqrt{1+2\check{a}n^{1/3}}$.

Since $|R_\lambda f(x)|$ is bounded, the above integral is bounded by an integral which is of a type which is known to go to zero exponentially, as $n \to \infty$.

Similarly

$$\left|\int_{\lambda_3}^{-\infty} e^{\lambda t} R_\lambda f(x)\, d\lambda\right|$$

goes to zero exponentially.

Thus lemma 5 is proved.

□

Proof of Lemma 6: Recall

$$\|R_\lambda^n f\|_2^2 = \sum_{j=1}^{n-1} \frac{|R_\lambda^n f(x_j)|^2}{a^n(x_j)} h$$

Thus

$$|R_\lambda^n f(x_j)| \le n \frac{\|R_\lambda^n f\|_2^2}{\min \{a\}} .$$

Also the following calculations shows

$$R_\lambda^n f(x) = \sum_{j=1}^{n-1} \frac{\langle f, Y_j^n \rangle Y_j^n(x)}{\lambda - \lambda_j^n}$$

where Y_j^n and λ_j^n are the normalized eigenfunctions and eigenvalues of A^n respectively. Consider

$$(\lambda - A^n) \sum_{j=1}^{n-1} \frac{\langle f, Y_j^n \rangle Y_j^n}{\lambda - \lambda_j^n} = \sum_{j=1}^{n-1} \frac{\langle f, Y_j^n \rangle (\lambda - A^n)}{\lambda - \lambda_j^n} Y_j^n$$

$$= \sum_{j=1}^{n-1} \frac{\langle f, Y_j^n \rangle (\lambda - \lambda_j^n) Y_j^n}{\lambda - \lambda_j^n} = f.$$

Now that the alternate formula for $R_\lambda^n f$ has been proved, observe that

$$\|R_\lambda^n f\| \leq \frac{1}{\min\{\lambda-\lambda_j\}} \|\sum_{j=1}^{n-1} \langle f, Y_j^n\rangle Y_j^n\|_2 \leq \|f\|_2,$$

for the discrete L_2 norm.

The last inequality follows from the choice of contour which gives $1 \leq |\lambda-\lambda_j|$. Now

$$|R_\lambda^n f(x_j)|^2 \frac{1}{a^n(x_j)} \frac{1}{n} \leq \|R_\lambda^n f\|_2^2.$$

So

$$|R_\lambda^n f(x_j)|^2 \leq n \max\{a^n\} \|f\|_2^2 .$$

The discrete L_2 norm of f is a Riemann Sum for the continuous L_2 norm of f. Hence there is a uniform bound for the discrete $\|f\|_2$ for all n. Thus $R_\lambda^n f(x_j) \leq \sqrt{n}\, K_3$ for some $K_3 > 0$. □

<u>Proof of Lemma 7</u>: Let $\lambda = u + iv$, with

$$-\frac{1}{\tau} - 1 \leq u \leq -2\check{a}n^{1/3},$$

and

$$|v| \leq 2\sqrt{1-u}.$$

Then,

$$-\tau \leq 1 + u\tau \leq 1 - 2\check{a}n^{1/3}\tau.$$

So,

$$|1+\lambda\tau| \leq \max\left\{\tau\ ,\ 1-2\check{a}\ n^{1/3}\ \tau\right\} + \max\left\{2\tau\sqrt{2+\frac{1}{\tau}}\ ,\ 2\tau\sqrt{2\check{a}\ n^{1/3}}\right\}$$

Recall $\tau = \dfrac{1}{4\|a^n\|_\infty n^2}$. Thus for large enough n, $|1+\lambda\tau| \leq 1 - \dfrac{c_1}{n^{5/3}}$ for some $c_1 > 0$. So,

$$|1+\lambda\tau|^k \leq \left(\left|1-\frac{c_1}{n^{5/3}}\right|^{n^{5/3}}\right)^{k/(n^{5/3})} ,$$

and as $n \to \infty$,

$$\left|1-\frac{c_1}{n^{5/3}}\right|^{n^{5/3}} \to e^{-c_1}.$$

Therefore, for n large enough

$$\left|1-\frac{c_1}{n^{5/3}}\right|^{n^{5/3}} < c_2 < 1, \quad \text{for some } c_2.$$

Recall $k = t/\tau = t\,4\|a^n\|_\infty n^2$. Thus $k/n^{5/3} = c_3 n^{1/3}$ for some $c_3 > 0$. Finally

$$|1+\lambda\tau|^k \le c_2^{c_3 n^{1/3}} .$$

The length of the contour of integration is bounded by a polynomial in n, say $L(n)$. Thus

$$\oint |(1+\lambda\tau)^k|\,|R_\lambda^n f(x)|\,|d\lambda| \le L(n)\, c_2^{c_3 n^{1/3}} K_3\sqrt{n}$$

after use of Lemma 6. Since $0 < c_2 < 1$ and $0 < c_3$, the above expression is damped by the exponential term. So, for n large enough,

$$\oint |1+\lambda\tau|^k |R_\lambda f(x)|\,|d\lambda| < K_4/n \quad \text{for some } K_4 > 0.$$

□

Proof of Lemma 8: Use of the following formula for the difference of a product verifies the equation

$$\Delta(p_j q_j) = q_{j+1}\Delta p_j + p_j \Delta q_j .$$

□

Proof of Lemma 9:

$$\Delta\Big(p(x_j)-q(x_j)\Big) = \Big(F\big(x_j, p(x_j)\big) - G\big(x_j, q(x_j)\big) - \big(p(x_j)-q(x_j)\big)\Big) =$$

$$= F\left(x_j,p(x_j)\right)-G\left(x_j,p(x_j)\right) + \left[\frac{G\left(x_j,p(x_j)\right)-G\left(x_j,q(x_j)\right)}{p(x_j)-q(x_j)} -1\right]\left(p(x_j)-q(x_j)\right)$$

Let

$$b_j = \left(F(x_j),p(x_j)\right) - G\left(x_j,p(x_j)\right)/h$$

and

$$c_j = \left[\frac{G\left(x_j,p(x_j)\right)-G\left(x_j,q(x_j)\right)}{p(x_j)-q(x_j)} -1\right]/h$$

Now use lemma 9 to get,

$$|p(x_j)-q(x_j)| \leq \left(|p(x_1)-q(x_1)| + \sum_{i=2}^{j} |b_{j-1}|\left|\prod_{k=2}^{i} (1+hc_{k-1})^{-1}\right|h\right)\prod_{i=2}^{j} |1+hc_{j-1}|.$$

Note that,

$$|b_{j-1}| \leq ||F-G||_\infty.$$

Also, $|(1+hc_{k-1})^{-1}| \leq ||(\frac{d}{dq}G)^{-1}||_\infty$. Thus

$$\left|\prod_{k=2}^{i} (1+hc)^{-1}\right| \leq m^{j-1} \quad .$$

Finally $\prod_{i=2}^{j} |1+hc_{j-1}| \leq ||\frac{d}{dq}G||_\infty^{j-1}$.

So

$$|p(x_j)-q(x_j)| \leq \left(|p(x_1)-q(x_1)|+m^{j-1}(j-1)\|F-G\|_\infty\right)\left\|\frac{\partial}{\partial q}G\right\|_\infty^{j-1}.$$

□

Proof of Lemma 10: This is just the vector form of lemma 9. The reasoning in lemma 9 and 10 needs to be adjusted to suit the vector forms of the statements.

In lemma 8, b_j is a vector and c_j is a matrix operating on the vector p_j.

$\prod_{k=2}^{i}(1+hc_{k-1})$ is a matrix operating on the vector b_{i-1}.

$\prod_{i=2}^{j}(1+hc_{j-1})$ is a matrix acting on the vector

$$\left[p_1+\sum_{i=2}^{j}\prod_{k=2}^{i}(1+hc_{k-1})b_{i-1}h\right].$$

The product formula for checking the vector form of lemma 9 becomes

$$\Delta(B_j q_j) = (\Delta B_j)q_{j+1} + B_j\Delta q_j$$

where B_j is a matrix acting on q_j.

The manipulations used to prove lemma 9 applies to the vector case if 1 is replaced by the identity matrix.

Thus lemma 10 is proved.

□

Proof of Lemma 11: Recall

$$\alpha_j = Y_0(\lambda,x_j)\ \sin\sqrt{\frac{-\lambda}{a_j}}\ x_j + \sqrt{\frac{a_j}{-\lambda}}\ \frac{d}{dx}\ Y_0(\lambda,x_j)\ \cos\sqrt{\frac{-\lambda}{a_j}}\ x_j,$$

$$\beta_j = Y_0(\lambda,x_j)\ \cos\sqrt{\frac{-\lambda}{a_j}}\ x_j + \sqrt{\frac{a_j}{-\lambda}}\ \frac{d}{dx}\ Y_0(\lambda,x_j)\ \sin\sqrt{\frac{-\lambda}{a_j}}\ x_j.$$

Also $\sqrt{-\lambda} = r + i$, $-\infty < r < \infty$ and $0 < a < a_j$. Furthermore $\sqrt{\lambda}\ Y_0(\lambda,x)$ and $\frac{d}{dx}\ Y_0(\lambda,x)$ are bounded (from lemma 2). Thus α_j and β_j are bounded. $\square$

Proof of Lemma 12: Lemma 10 will be used. Let $F\left(x_j,\begin{pmatrix}\alpha_j\\ \beta_j\end{pmatrix}\right)$ be the recursion relation taking

$$\begin{pmatrix}\alpha_j\\ \beta_j\end{pmatrix}\ \text{to}\ \begin{pmatrix}\alpha_{j+1}\\ \beta_{j+1}\end{pmatrix}.$$

Let $F^n\left(x_j,\begin{pmatrix}\alpha_j^n\\ \beta_j^n\end{pmatrix}\right)$ be the recursion relation taking $\begin{pmatrix}\alpha_j^n\\ \beta_j^n\end{pmatrix}$ to $\begin{pmatrix}\alpha_{j+1}^n\\ \beta_{j+1}^n\end{pmatrix}$.

The expression of F^n at the end of section 2 is not immediately seen to be close to F. However, if the correct terms are added and subtracted to change $x_j + \frac{h}{2}$ to x_j, and then formulas for the cos and sin of a difference are applied, then F^n takes the following form

$$\alpha_{j+1}^{n} = \Bigg[\alpha_j^n \cos(x_j[u_{j+1}-u_j])+\beta_j^n \sin(x_j[u_{j+1}-u_j])$$

$$+\left(\sqrt{\frac{a_j+1}{a_j}}-1\right)(\alpha_j^n \cos u_jx_j-\beta_j^n \sin u_jx_j)\cos u_{j+1}x_j$$

$$+(\alpha_j^n \sin u_jx_j+\beta_j^n \cos u_jx_j)\,\Delta_{\frac{h}{2}}\,\sin u_{j+1}x_j$$

$$+\sqrt{\frac{a_{j+1}}{a_j}}\cos u_{j+1}x_j\,\Delta_{\frac{h}{2}}\,(\alpha_j^n \cos u_jx_j-\beta_j \sin u_jx_j)\Bigg]\Big/\sqrt{1+\frac{h^2\lambda}{4a_{j+1}}}\,,$$

$$\beta_{j+1}^{n} = \Bigg[\alpha_j^n \sin(x_j[u_{j+1}-u_j])+\beta_j^n \cos(x_j[u_{j+1}-u_j])$$

$$+\left(1-\sqrt{\frac{a_{j+1}}{a_j}}\right)(\alpha_j^n \cos u_jx_j-\beta_j^n \sin u_jx_j)\sin u_{j+1}x_j$$

$$+(\alpha_j^n \sin u_jx_j+\beta_j^n \cos u_jx_j)\,\Delta_{\frac{h}{2}}\,\cos u_{j+1}x_j$$

$$-\sqrt{\frac{a_{j+1}}{a_j}}\sin u_{j+1}x_j\,\Delta_{\frac{h}{2}}\,(\alpha_j^n \cos u_jx_j-\beta_j^n \sin u_jx_j)\Bigg]\Big/\sqrt{1+\frac{h^2\lambda}{4a_{j+1}}}\,.$$

In the above expression $\Delta_{\frac{h}{2}}$ is the difference operator defined by

$$\Delta_{\frac{h}{2}}\,g(x) = g(x+\tfrac{h}{2}) - g(x).$$

It will be shown that $\|F^n-F\|_\infty \le ch^2$ for some $c > 0$. First note that the first two lines in the last expression of α_{j+1}^n and β_{j+1}^n are nearly equal to the expressions for α_{j+1} and β_{j+1}. The

contribution of $\sqrt{1+\frac{h^2\lambda}{4a_j}}$ in the denominator to the difference of F^n and F is of order $h^2\lambda$ if $\left|\frac{h^2\lambda}{4a_j}\right|$ is bounded away from 1. However,

$$|\lambda| \leq 2\check{a}n^2 + 2\sqrt{1+2\check{a}n^2} .$$

So,

$$\left|\frac{h^2\lambda}{4a_j}\right| \leq \frac{1}{2} + \frac{h}{\sqrt{a_j}} \sqrt{h^2+2} < 1,$$

for h small enough.

Next it will be shown that $\left|u_j - \sqrt{\frac{-\lambda}{a_j}}\right|$ is of order $h^2\sqrt{|\lambda|}$. Recall $u_j = \frac{2}{h} \sin^{-1} \frac{h}{2}\sqrt{\frac{-\lambda}{a_j}}$. Taylor series expansion gives

$$|y - \frac{1}{\varepsilon} \sin^{-1}\varepsilon y| \leq \frac{\varepsilon^2|y|^3}{2(1-\varepsilon^2|y^2|)^{3/2}}$$

for $\varepsilon > 0$ and $|\varepsilon y| < 1$. Thus

$$\left|u_j - \sqrt{\frac{-\lambda}{a_j}}\right| \leq \frac{h^2\sqrt{|\lambda|/a_j}}{8\left(1-\frac{h^2|\lambda|}{4a_j}\right)^{3/2}} .$$

and we have already seen that $\left|\frac{h^2\lambda}{4a_j}\right|$ is bounded below 1.

So $\left|u_j - \sqrt{\frac{-\lambda}{a_j}}\right| \le ch^2\sqrt{|\lambda|}$ for some $c > 0$. So far

$$\left|F^n\left(x_j,\begin{pmatrix}\alpha_j\\ \beta_j\end{pmatrix}\right)-F\left(x_j,\begin{pmatrix}\alpha_j\\ \beta_j\end{pmatrix}\right)\right| \le ch^2|\lambda|(|\alpha|+|\beta|)$$

for some $c > 0$, if the first two lines of F^n are compared to F.

The last two lines of the expressions for α^n_{j+1} and β^n_{j+1} will be seen to be small of order $h^2|\sqrt{\lambda}|(|\alpha^n|+|\beta^n|)$ as shown by this sample calculation.

Consider the last two lines of the expression for α^n_{j+1}.

$$(\alpha^n_j \sin u_j x_j+\beta^n_j \cos u_j x_j) \,\triangle\tfrac{h}{2}\, \sin u_{j+1}x_j$$

$$+ \sqrt{\frac{a_{j+1}}{a_j}} \cos u_{j+1}x_j \,\triangle\tfrac{h}{2}\, (\alpha^n_j \cos u_j x_j-\beta^n_j \sin u_j x_i)$$

$$= (\alpha^n_j \sin u_j x_j+\beta^n_j \cos u_j x_j) 2 \sin \frac{u_{j+1}h}{4} \cos u_{j+1}(x_j+\frac{h}{4})$$

$$+ \cos u_{j+1}x_j\left(-\alpha^n_j \sin u_j(x_j+\frac{h}{4})-\beta^n_j \cos u_j(x_j+\frac{h}{4})\right)2 \sin \frac{u_{j+1}h}{4} .$$

The above expression is the difference of two nearly equal terms. Note that

$$|\sin u_j x_j - \sin u_j(x_j+\frac{h}{4})| \le c_1 h|u_j| \quad \text{for some } c_1 > 0,$$

$$|\cos u_j x_j - \cos u_j(x_j+\frac{h}{4})| \le c_2 h\, u_j \quad \text{for some } c_2 > 0,$$

$$\left|\cos u_{j+1}(x_j+\frac{h}{4}) - \cos u_{j+1}x_j\right| \le c_3 h|u_{j+1}| \quad \text{for some} \quad c_3 > 0,$$

$$2\left|\sin \frac{u_{j+1}h}{4}\right| \le c_4 h|u_{j+1}| \quad \text{for some} \quad c_4 > 0,$$

and

$$|u_j| < c_5\sqrt{|\lambda|} \qquad\qquad c_5 > 0.$$

Thus the terms being considered are small of order $h^2\sqrt{|\lambda|}\,(|\alpha_j^n|+|\beta_j^n|)$.

Recall that lemma 12 says that $(|\alpha_j|+|\beta_j|)$ is bounded for $1 \le j \le n$. So

$$\left|F^n\left(x_j,\begin{pmatrix}\alpha_j\\ \beta_j\end{pmatrix}\right)-F\left(x_j,\begin{pmatrix}\alpha_j\\ \beta_j\end{pmatrix}\right)\right| \le h^2 c_6|\lambda| \quad \text{for some} \quad c_6 > 0.$$

This gives

$$(j-1)\,||F^n-F|| \le hc_6|\lambda|.$$

Now consider the Jacobian of $F(x,\cdot)$. Observe that $F(x,\cdot)$ is a linear operator on $\begin{pmatrix}\alpha_j\\ \beta_j\end{pmatrix}$ and its matrix form is given below.

$$\begin{pmatrix} \cos - \sqrt{-\lambda}\, x_j\, \Delta\, \frac{1}{a_j} & -\sin - \sqrt{-\lambda}\, x_j\, \Delta\, \frac{1}{a_j} \\ \sin - \sqrt{-\lambda}\, x_j\, \Delta\, \frac{1}{a_j} & \cos - \sqrt{-\lambda}\, x_j\, \Delta\, \frac{1}{a_j} \end{pmatrix}$$

$$+\frac{\Delta\sqrt{a_j}}{\sqrt{a_j}}\begin{pmatrix} \cos -\sqrt{\frac{-\lambda}{a_{j+1}}}\, x_j & 0 \\ 0 & \sin -\sqrt{\frac{-\lambda}{a_{j+1}}}\, x_j \end{pmatrix}\begin{pmatrix} \cos\sqrt{\frac{-\lambda}{a_j}}\, x_j & -\sin\sqrt{\frac{-\lambda}{a_j}}\, x_j \\ \cos\sqrt{\frac{-\lambda}{a_j}}\, x_j & -\sin\sqrt{\frac{-\lambda}{a_j}}\, x_j \end{pmatrix}$$

The first term in the above expression is a rotation matrix with angle dominated by $\Delta \frac{1}{a_j}$. Also recall that the imaginary part of $\sqrt{-\lambda}$ is bounded. Thus the first term is $I + hc_j$ where $|c_j|$ is bounded, and I is the identity matrix. Also the second term of F can be expressed as hE_j with E_j bounded. So $JF = (I+hE)$ with $||E||$ bounded. Using the notation of lemma 10, observe

$$m^{j-1} \leq (1-h||E||)^{-n}$$

and

$$||JF||^{j-1} \leq (1+h||E||)^{n}.$$

Since $(1-||E||h)^{-n}$ and $(1+||E||h)^{n}$ converge to $e^{||E||}$, they are bounded. Now consider

$$\left|\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix} - \begin{pmatrix}\alpha_1^n\\ \beta_1^n\end{pmatrix}\right| .$$

$$\alpha_1 - \alpha_1^n = \sqrt{\frac{a_1}{-\lambda}}\left(1-\frac{1}{\sqrt{1+\frac{h^2\lambda}{4a_1}}}\right).$$

Thus $|\alpha_1-\alpha_1^n| < ch^2\lambda$ for some $c > 0$.

Also $\beta_1 - \beta_1^n = 0$. So

$$\left|\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}-\begin{pmatrix}\alpha_1^n\\ \beta_1^n\end{pmatrix}\right| < ch^2\lambda \quad \text{for some} \quad c>0.$$

Now lemma 10 yields the statement of lemma 12 (i). Symmetry proves lemma 12 (ii). □

<u>Proof of Lemma 13</u>: Recall $|R_\lambda f(x)|$ is bounded from lemma 4. Thus, using the symmetry of the contour, it is enough to show

$$\int_1^{\lambda_0} |e^{\lambda t}-(1+\lambda\tau)^k||d\lambda| < \frac{K}{n} \quad \text{for some} \quad K_5 > 0.$$

Observe $e^{\lambda t} = (e^{\lambda\tau})^k$. Then recall the factorization

$$p^k - q^k = (p-q) \sum_{\ell=0}^{k-1} p^{k-1-\ell} q^\ell,$$

which gives

$$|p^k-q^k| \leq |p-q| k \max\{|p|^{k-1}, |q|^{k-1}\}.$$

Now

$$\int_0^{\lambda_0} |e^{\lambda t}-(1+\lambda\tau)^k||d\lambda| \le \int_0^{\lambda_0} k|e^{\lambda\tau}-(1+\lambda\tau)| \max\{|e^{\lambda\tau}|^{k-1}, |1+\lambda\tau|^{k-1}\}|d\lambda|.$$

Recall $\lambda = (1-r^2) - i\,2r$, also for λ on the contour

$$|r| \le \sqrt{1+2\overset{\vee}{a}n^{1/3}}.$$

Thus

$$|\lambda\tau| \le \frac{2}{4\|a^n\|_\infty n^2} + \frac{2\overset{\vee}{a}}{4\|a^n\|_\infty n^{1/3}} + \frac{2\sqrt{1+2\overset{\vee}{a}n^{1/3}}}{4\|a^n\|_\infty n^2}.$$

So at least

$$|\lambda\tau| < c_1 < 1.$$

Also, for some $c_2 > 0$

$$|\ell n\,(1+\lambda\tau) - \lambda\tau| \le \frac{|\lambda^2\tau^2|}{2(1-|\lambda\tau|^2)} < c_2 .$$

Thus,

$$|1+\lambda\tau| = |e^{\ell n(1+\lambda\tau)}| \le e^{c_2}|e^{\lambda\tau}|$$

$$|e^{\lambda\tau}-(1+\lambda\tau)| \le \frac{1}{2}|\lambda\tau|^2,$$

$$|d\lambda| \le |2r+2||dr|,$$

and $k = t/\tau \leq c_3 n^2$ for some $c_3 > 0$. So,

$$\int_0^{\lambda_0} |e^{\lambda t} - (1+\lambda\tau)^k||d\lambda| \leq n^2 c_3 \int_0^{r_0} |r+1||\lambda\tau|^2 e^{c_2}|e^{\lambda\tau}||dr|,$$

where $r_0 = -\sqrt{1+2\check{a}n^{1/3}}$. Observe,

$$|e^{\lambda\tau}| = e^{(1-r^2)\tau} \leq c_4 \quad \text{for some } c_4 > 0.$$

Also,

$$|r+1||\lambda\tau|^2 \leq (c_5|r|^5 + c_6)/n^4$$

for some $c_5, c_6 > 0$. Thus

$$\int_0^{\lambda_0} |e^{\lambda t} - (1+\lambda\tau)^k||d\lambda| \leq c_7 \frac{1}{n^2} \int_0^{|r_0|} (c_5|r|^5 + c_6)\,dr ,$$

for some $c_7 > 0$. At this point the choice of λ_0 and λ_3 becomes important. Observe $|r_0| \leq c_8 n^{1/6}$ for large n. Thus

$$\int_0^{\lambda_0} |e^{\lambda t} - (1+\lambda\tau)^k||d\lambda| \leq \frac{c_7}{n^2}\left[\frac{c_5}{6} r^6 + c_6 r\right]_0^{c_8 n^{1/6}}$$

$$\leq \frac{K_6}{n} \quad \text{for some } K_6 > 0.$$

□

Section Five

In this section the proof of the main result on page 11 will be completed. As previously noted there is nothing to prove if $z = 1$. Here it will be shown that for $0 < t$ and $z \in [0,1)$,

$$\left| \oint e^{\lambda t} R_\lambda f(x) d\lambda - \oint (1+\lambda\tau)^k R_\lambda^n f(x) d\lambda \right| < K/n$$

for some $K > 0$.

From lemma 4, the above integrals exist.

First define three contours,

$$\Gamma_1 = \{\lambda\colon \lambda=(1-r^2)+i2r,\ |r| \leq \sqrt{1+2\check{a}n^{1/3}}\ \},$$

$$\Gamma_2 = \{\lambda\colon \lambda=(1-r^2)+i2r,\ \sqrt{1+2\check{a}n^{1/3}} \leq |r|\},$$

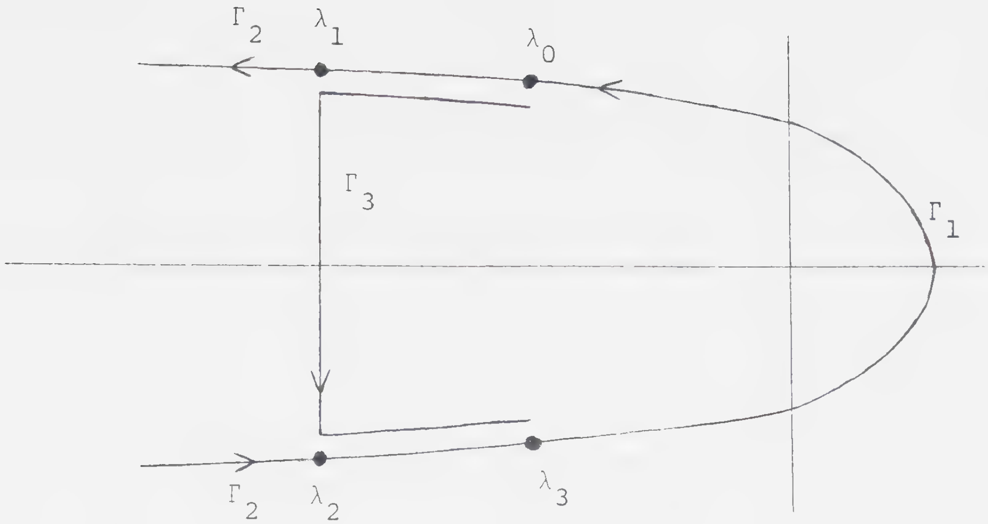
$$\Gamma_3 = \{\lambda\colon \lambda=(1-r^2)+i2r,\ \sqrt{1+2\check{a}n^{1/3}} \leq |r| \leq \sqrt{2+\frac{1}{\tau}},$$

$$\text{or } \lambda = -\frac{1}{\tau} - 1 + iv \text{ with } |v| \leq \sqrt{2+\frac{1}{\tau}}\ \}.$$

Γ_1 is oriented counter clockwise.

Γ_2 is disconnected with the top portion oriented to the left and the bottom to the right.

Γ_3 is oriented counter clockwise.



Lemma 5 and 7 show that

$$\left|\int_{\Gamma_2} e^{\lambda t} R_\lambda f(x)\, d\lambda\right| < K_2/n$$

and

$$\left|\int_{\Gamma_3} (1+\lambda\tau)^k R_\lambda^n f(x)\, d\lambda\right| < K_4/n.$$

Thus it is enough to show that

$$\left|\int_{\Gamma_1} \left(e^{\lambda t} R_\lambda f(x) - (1+\lambda\tau)^k R_\lambda^n f(x)\right) d\lambda\right| < K/n \quad \text{for some} \quad n > 0.$$

To this end it is enought that the following three statements are true:

i) $\left|\int_{\Gamma_1} \left(e^{\lambda t} - (1+\lambda\tau)^k\right) R_\lambda f(x)\, d\lambda\right| < K/n \quad \text{for some} \quad K > 0$

ii) $\quad \left|\int_{\Gamma_1} e^{\lambda t}\left(R_\lambda f(x)-R_\lambda^n f(x)\right)d\lambda\right| < K/n \quad$ for some $K > 0$

iii) $\quad \left|\int_{\Gamma_1} \left(e^{\lambda t}-(1+\lambda\tau)^k\right)\left(R_\lambda f(x)-R_\lambda^n f(x)\right)d\lambda\right| < K/n \quad$ for some $K > 0$

Lemma 13 proves (i).

To prove (ii) recall

$$R_\lambda f(x) = \int_0^1 G_\lambda(x,y)f(y)d\lambda.$$

Thus $R_\lambda f(x)$ is approximated by the sum

$$\sum_{j=1}^{n-1} G_\lambda(x,y_j)f(y_j)h.$$

Recall $\quad G_\lambda(x,y) = \dfrac{1}{a^n(y)w(\lambda)}\begin{cases} Y_1(\lambda,y)Y_0(\lambda,x), & x \le y \\ Y_0(\lambda,y)Y_1(\lambda,x), & y \le x. \end{cases}$

From lemma 2 and since $\dfrac{1}{a^n(y)}$ has bounded variation uniformly in n, $G(x,\cdot)f(\cdot)$ has uniform bounded variation. Thus

$$\left|R_\lambda f(x)-\sum_{j=1}^{n-1} G_\lambda(x,y_j)f(y_j)h\right| \le c_1 h$$

for some $c_1 > 0$.

Also recall

$$R_\lambda^n f(x) = \sum_{j=1}^{n-1} G_\lambda^n(x,y_j) f(y_j) h$$

where $$G_\lambda^n(x,y) = \frac{1}{a^n(y) w^n(\lambda)} \begin{cases} Y_1^n(\lambda,y) Y_0^n(\lambda,n), & x \le y \\ \\ Y_0^n(\lambda,y) Y_1^n(\lambda,x), & y \le x. \end{cases}$$

Furthermore

$$w(\lambda) = Y_0(\lambda,1) \quad \text{and} \quad w^n(\lambda) = Y_0^n(\lambda,1).$$

Lemma 12 together with lemma 3 gives

$$|G_\lambda(x,y) - G_\lambda^n(x,y)| \le |\lambda|^{5/2} c_2/n$$

for some $c_2 > 0$, and large enough n. Thus

$$|R_\lambda f(x) - R_\lambda^n f(x)| \le |\lambda|^{5/2} c_3/n.$$

Finally

$$\left| \int_{\Gamma_1} e^{\lambda t} \left(R_\lambda f(x) - R_\lambda^n f(x) \right) d\lambda \right| \le \frac{c_3}{n} \int_{\Gamma_1 \cup \Gamma_2} e^{\lambda t} |\lambda|^{5/2} d\lambda .$$

Since the last integral is finite (ii) is proved. To prove (iii)

it is enough to show

$$\int_{\Gamma_1} \left(|e^{\lambda t}| + |1+\lambda\tau|^k\right) |R_\lambda f(x) - R_\lambda^n f(x)|\,|d\lambda| \le K/n$$

for some $K > 0$.

Notice that for some $c_4 > 0$

$$|k \,\ell n\, (1+\lambda\tau) - k\lambda\tau| \le \frac{|k\lambda^2\tau^2|}{2(1-|\lambda\tau|^2)} \le \frac{|t\lambda^2\tau|}{2(1-|\lambda\tau|^2)} \le \lambda t\, c_4 ,$$

because $\lambda\tau$ is bounded on Γ_1. Thus

$$|1+\lambda\tau|^k \le |e^{k\,\ell n(1+\lambda\tau)}| \le e^{\lambda t\, c_4} |e^{k\lambda\tau}| \le c_5 |e^{\lambda t}|$$

for some $c_5 > 0$ and all λ on Γ_1.

So

$$\int_{\Gamma_1} \left(|e^{\lambda t}| + |1+\lambda\tau|^k\right) |R_\lambda f(x) - R_\lambda^n f(x)|\,|d\lambda| \le \int_{\Gamma_1} (1+c_5)|e^{\lambda t}|\,|\lambda|^{5/2} c_3 \frac{1}{n} |d\lambda|$$

$$\le \frac{c_6}{n} \int_{\Gamma_1 \cup \Gamma_2} |e^{\lambda t}|\,|\lambda|^{5/2} |d\lambda| .$$

This last integral is finite, hence (iii) is proved. This completes the proof of the main result.

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APPENDIX

This appendix contains results of two types. The first type of theorem would be standard if some restrictive conditions were imposed. An outline of proof is given which hurries over the familiar arguments.

The second type of result found here is a direct consequence of some very difficult and lengthy work by others (for example, pointwise convergence theory, and harmonic analysis of the sort due to Elias Stein). Such results are proved here in a form sufficiently reduced in generality that elementary arguments apply.

The first section deals with the spectral theory of Sturm-Liouville operators. The second section solves boundary value problems using the first section, and the third section proves the existence of certain diffusions, used in section 2. The semi-group point of view is also briefly discussed in section 2.

§1. A class of Sturm-Liouville operators

Let $a(\cdot)$ be a Borel-measurable function satisfying $0 < k_1 \leq a(\cdot) \leq k_2$. Write $\|f\|_2 = \{\int_0^1 a(x)^{-1}|f(x)|^2 dx\}^{\frac{1}{2}}$, and define L_a^2 as the space of complex functions whose norm is finite. Then L_a^2 becomes a Hilbert space with the inner product

$$\langle f,g\rangle = \int_0^1 a(x)^{-1} f(x)\overline{g(x)}\,dx, \quad f,\ g \in L_a^2.$$

Define D_2 to be the class of funtions f for which:

(i) $f \in C^1[0,1]$;

(ii) $f' \in AC[0,1]$,

(iii) $a(\cdot)f''(\cdot) \in L_a^2$,

(iv) $f(0) = 0 = f(1)$.

By definition, D_2 is the domain of the operator A_2 mapping D_2 into L_a^2, where $(A_2 f)(\cdot) = a(\cdot)f''(\cdot)$, almost everywhere defined.

At this stage we omit proof that D_2 is dense in L_a^2, as we later need a stronger result (cf. 3.2).

1.1 Lemma Given $\lambda \in \mathbb{C}$, there are unique functions $Y_0(\lambda,\cdot)$, $Y_1(\lambda,\cdot)$ satisfying:

(i) $Y_0(\lambda,\cdot)$, $Y_1(\lambda,\cdot) \in C^1[0,1]$;

(ii) $Y_0'(\lambda,\cdot)$, $Y_1'(\lambda,\cdot) \in AC[0,1]$;

(iii) $a(\cdot)Y_0''(\lambda,\cdot) = \lambda Y_0(\lambda,\cdot)$, $a(\cdot)Y_1''(\lambda,\cdot) = \lambda Y_1(\lambda,\cdot)$;

(iv) $Y_0(\lambda,0) = 0$, $Y_0'(\lambda,0) = 1$; $Y_1(\lambda,1) = 0$, $Y_1'(\lambda,1) = -1$.

Proof is standard, using the contraction mapping theorem in $C[\alpha,\beta]$ to obtain local existence first.

□

Spectrum, Resolvent Operators, and Greens Functions

Write $w(\lambda) = Y_0(\lambda,1)$ and note that $w(\lambda) = 0$ iff λ is an eigenvalue of A_2, with eigenfunction $Y_0(\lambda,\cdot)$. Since A_2 is a symmetric operator its eigenvalues must be real, and its eigenspaces mutually orthogonal. Separability of L_a^2 forces the set of eigenvalues to be countable, and obviously each eigenspace is one-dimensional. Clearly 0 is not an eigenvalue, and yet the eigenvalues must be non-positive because $\langle A_2 f,f\rangle \leq 0$, all $f \in D_2$. (Proof uses integration by parts.) Although the following lemma is not needed for the proofs which follow, it is used in Chapters 2,3, and simplifies our presentation now.

1.2 Lemma (cf. [8] p. 401)

The eigenvalues of A_2 form a decreasing sequence $\{\lambda_j | j=1,2,\ldots\} \subseteq (-\infty,0)$, satisfying $|\lambda_j| \geq C j^2$ for some $C > 0$.

Write Y_j for the normalized eigenfunction associated with λ_j, so

$$Y_j(\cdot) = \|Y_0(\lambda_j,\cdot)\|_2^{-1} Y_0(\lambda_j,\cdot).$$

For $\lambda \notin \{\lambda_1,\lambda_2,\ldots\}$, define

$$G_\lambda(x,y) = \begin{cases} a(y)^{-1}w(\lambda)^{-1}Y_0(\lambda,x)Y_1(\lambda,y), & x \le y; \\ a(y)^{-1}w(\lambda)^{-1}Y_0(\lambda,y)Y_1(\lambda,x), & x \ge y. \end{cases}$$

Define the <u>resolvent operator</u> $R_\lambda : L_a^2 \to L_a^2$ by:

$$(R_\lambda f)(x) = \int_0^1 G_\lambda(x,y)f(y)dy, \qquad f \in L_a^2.$$

1.3 <u>Lemma</u> The operator R_λ maps L_a^2 onto D_2, and equals $(\lambda - A_2)^{-1}$.

<u>Proof</u>: (Outline) Writing out $(R_\lambda f)(x)$ as the sum of two integrals, one may differentiate directly to show $R_\lambda f \in D_2$, and $(\lambda - A_2)R_\lambda f = f$. Equality of D_2 with its subspace $R_\lambda L_a^2$ follows from the assumption that $\lambda - A_2$ is 1:1 on D_2.

$\square$

1.4 <u>Theorem</u> Let $f \in L_a^2$, and write

$$\hat{f}_j = \langle f, Y_j \rangle, \quad j = 1,2,\ldots .$$

Then

(i) $f = \sum_{j=1}^{\infty} \hat{f}_j Y_j$;

(ii) $R_\lambda f = \sum_{j=1}^{\infty} (\lambda - \lambda_j)^{-1} \hat{f}_j Y_j$;

where the series converge in L_a^2.

Proof: (Outline) It is easy and standard to show R_λ is compact. Also, one can see that the adjoint of R_λ is associated with $\overline{G_\lambda(\cdot,\cdot)}$, and this leads to the conclusion that R_λ is a normal operator. The spectral theorem for a compact normal operator (cf. [14], p. 345) now yields the theorem.

□

1.5 Corollary

(i) A function g is in D_2 iff

$$\sum_{j=1}^{\infty} |\lambda-\lambda_j|^2 |\hat{g}_j|^2 < \infty.$$

(ii) For $g \in D_2$,

$$A_2 g = \sum_{j=1}^{\infty} \lambda_j \hat{g}_j Y_j \quad (L_a^2 \text{ convergence}).$$

Proof: (i) This follows from 1.4 using the fact that D_2 is the range of R_λ, and the fact that $f \in L_a^2$ iff $\sum_{j=1}^{\infty} |\hat{f}_j|^2 < \infty$.

(ii) Apply R_λ to the series on the right, noting that it converges because of (i). Re-organization yields (ii).

□

§2. Solution of boundary-value problems using eigenfunction expansions

2.1 Theorem Let $a(\cdot)$ be measurable on $[0,1]$, with $k_1 \le a(\cdot) \le k_2$. Let $f \in L_a^2[0,1]$. Then there exists a function $\phi(\cdot,\cdot)$ continuous on the set $[0,1] \times (0,\infty)$ satisfying:

1) $\phi(0,t) = \phi(1,t) = 0, \quad t \ge 0;$

2) $\lim_{t\to 0^+} \|\phi(\cdot,t)-f(\cdot)\|_2 \to 0;$

3) $\frac{\partial\phi}{\partial t}$ is continuous on $[0,1] \times (0,\infty)$;

4) $\frac{\partial\phi}{\partial x}$ is absolutely continuous on $[0,1]$, $t > 0$;

5) $a(\cdot)\frac{\partial^2\phi}{\partial x^2} = \frac{\partial\phi}{\partial t}$ almost all x, for $t > 0$.

6) If $f \in C_0[0,1]$, then $\lim_{t\to 0^+} \|\phi(\cdot,t)-f(\cdot)\|_\infty \to 0$.

7) $\phi(x,t) = \sum_{j=1}^{\infty} e^{\lambda_j t} Y_j(x)\hat{f}_j,$

the series converging uniformly on $[0,1]$, for $t > 0$, where $\hat{f}_j = \langle f, Y_j\rangle$. Note that 7) includes uniqueness of $\phi(\cdot,\cdot)$.

Proof: Consider the series of 7). It was proved in Chapter 3 that $\|Y_j\|_\infty$ is bounded by a power of j. It is also a standard result (lemma 1.2) that $-\lambda_j \ge Cj^2$, for some $C > 0$. This proves $\sum_{j=1}^{\infty} e^{\lambda_j t}\|Y_j\|_\infty|\hat{f}_j| < \infty$, and hence the series of 7) converges uniformly on any rectangle $[0,1] \times [\varepsilon,\infty)$, $\varepsilon > 0$. The resulting function ϕ is continuous on $[0,1] \times (0,\infty)$. The same argument works for $\frac{\partial\phi}{\partial t}$ and justifies term by term differentiation, yielding:

$$\frac{\partial\phi}{\partial t}(x,t) = \sum_{j=1}^{\infty} e^{\lambda_j t}\lambda_j Y_j(x)\hat{f}_j, \quad t > 0.$$

Since $a(\cdot)Y_j''(\cdot) = \lambda_j Y_j(\cdot)$ a.e. on $[0,1]$, the series $a(\cdot)\sum_{j=1}^{\infty} e^{\lambda_j t} Y_j''(\cdot)\hat{f}_j$ converges in the norm of $L^{\infty}[0,1]$, and it then follows that the series is almost everywhere equal to $a(\cdot)\dfrac{\partial^2\phi}{\partial x^2}$, $t > 0$. The fact that the solution must be given by 7) follows by integrating 5) against Y_j.

Properties 2), 6) remain to be proved. For 2) note that:

$$\|\phi(\cdot,t)-f(\cdot)\|_2^2 = \sum_{j=1}^{\infty}\left(e^{\lambda_j t}-1\right)^2\hat{f}_j \to 0 \quad \text{as} \quad t \to 0^+.$$

Property 6) is considerably more subtle, as the series for f need not converge uniformly. Its proof will be included in the next section on semi-groups.

□

2.2 Semi-groups acting on $C[0,1]$

Let $f \in C_0[0,1]$, and write $T_t f$ for the solution $\phi(\cdot,t)$ of the BVP of the previous theorem, when $t > 0$. Write $T_0 = I$, and note that uniqueness of the solution to such a BVP implies that $T_{t+s} = T_t T_s$. We will prove that $\{T_t\}$ is a strongly continuous contraction semi-group on $C_0[0,1]$, in the following standard sense:

1) $T_{t+s} = T_t T_s$, $t,s \geq 0$,

2) $T_0 = I$,

3) $\|T_t\| \leq 1$, $t \geq 0$,

4) $\lim_{t\to 0}\|T_t f - f\| = 0$, all $f \in C_0[0,1]$.

Henceforth $\|\cdot\|$ will denote $\|\cdot\|_{\infty}$. Properties 1), 2) have already been noted. Property 4) is the (as yet unproved) assertion 6) of the previous theorem. In order to construct a Markov Process from $\{T_t\}$ we need properties 5), 6) below. Property 5) is obvious, and

property 6) will be proved with 3).

5) $T_t 1 = 1$;

6) $T_t f \geq 0$ whenever $f \geq 0$.

Proof of 3), 6): If g is a function on $(0,\infty)$, define $\underline{D}_t g = \liminf_{h\to 0^+} \frac{g(t+h)-g(t)}{h}$. Write F_t for the closed subset of $[0,1]$ on which $T_t f$ assumes its minimum. Write $\phi(x,t) = (T_t f)(x)$, $t > 0$. We claim that for $t > 0$,

$$\underline{D}_t \inf_x \phi(x,t) = \inf\{\frac{\partial\phi}{\partial t}(x,t) \mid x \in F_t\}. \tag{*}$$

This is an easy consequence of continuity of ϕ on $[0,1] \times (0,\infty)$. We can now prove 6) by contradiction. If $\inf_x \phi(x,t)$ is ever negative, then at some time t, $\underline{D}_t \inf_x \phi(x,t) < 0$, and $\inf_x \phi(x,t) < 0$. But then (*) shows $\frac{\partial\phi}{\partial t}(x_0,t)$ must be negative for some $x_0 \in F_t$, and hence negative on an interval about x_0. Since $a(x)\frac{\partial^2\phi}{\partial x^2}(x,t) = \frac{\partial\phi}{\partial t}(x,t)$ a.e., we conclude $\frac{\partial^2\phi}{\partial x^2}(x,t)$ is negative a.e. in an interval about x_0, and this violates the assumption $x_0 \in F_t$. The contradiction proves $\inf_x \phi(x,t)$ is never negative, which is 6).

As in the previous paragraph, we can show that for $t > 0$,

$$\overline{D}_t \sup_x \phi(x,t) = \sup\{\frac{\partial\phi}{\partial t}(x,t) \mid x \in G_t\},$$

where G_t is the set where ϕ assumes its maximum, and $\overline{D}_t$ denotes the upper derivate. An analogous argument shows that $\overline{D}_t \sup_x \phi(x,t)$

must always be ≤ 0, and hence $\sup_x \phi(x,t)$ is non-increasing with time. This proves 3).

Proof of 4): Because the operator norms $\|T_t\|$, $t > 0$ are uniformly bounded, we can show that the functions f for which 4) holds form a closed subspace of $C_0[0,1]$. It is easy to show that if f is one of the eigenfunctions Y_j, then 4) holds, because $T_t Y_j = e^{\lambda_j t} Y_j$. It remains only to show that the closed linear space generated by the eigenfunctions is all of $C_0[0,1]$. To prove this, use the Green's function representation of R_λ, to show that $\|R_\lambda f\|_\infty \leq K\|f\|_2$. This means R_λ is continuous as an operator from L_a^2 to $C_0[0,1]$. Since the linear span of the eigenfunctions is dense in L_a^2, and is mapped to itself by R_λ, the linear span of the eigenfunctions is uniformly dense in the range of R_λ. But we already showed that the range of R_λ is D_2, so the final step is to show D_2 is uniformly dense in $C_0[0,1]$, which is a consequence of Lemma 3.2. □

Extension of T_t to $C[0,1]$

If $f \in C[0,1]$, then Theorem 2.1 still holds provided (1) is replaced by (1') $\phi(0,t) = f(0)$, $\phi(1,t) = f(1)$, $t \geq 0$. To see this we need only solve the problem for the two extra functions 1 and x as candidates for f. In each case, $\phi(x,t) = f(x)$ all $t \geq 0$ solves the problem. This proves properties 1)...4) stated earlier just for $C_0[0,1]$.

§3. Existence of diffusions

Suppose $0 < k_1 \le a(\cdot) \le k_2$ as before, and make the additional assumption that the set N of discontinuities of $a(\cdot)$ has measure zero (i.e. $a(\cdot)$ is Riemann integrable).

3.1 Theorem There exists a transition function $P_\cdot(\cdot,\cdot)$ having the properties:

(o) $\lim_{t\to 0^+} t^{-1}P_t(x,[0,x-\delta] \cup [x+\delta,1]) = 0$, for all $\delta > 0$, $x \in [0,1]$;

(i) $\lim_{t\to 0^+} t^{-1} \int_{|y-x|<\delta} (y-x)P_t(x,dy) = 0$ for $\delta > 0$, $x \in [0,1]$

(ii) $\lim_{t\to 0^+} t^{-1} \int_{|y-x|<\delta} (y-x)^2 P_t(x,dy) = 2a(x)$ for $\delta > 0$, $x \notin N$;

(iii) $P_t(0,\{0\}) = 1 = P_t(1,\{1\})$, $t \ge 0$;

(iv) $P_\cdot(\cdot,\cdot)$ generates a strongly continuous semi-group acting on $C[0,1]$ by the formula:

$$(T_tf)(x) = \int f(y)P_t(x,dy).$$

The rest of the section will outline the proof of Theorem 3.1. Theorem 3.1 has essentially been proved (in a different form) in a very general context by Stroock and Varadhan (cf. [13], Chapter 3), but the proof is extremely long and hard. Because the spectral theory just developed is necessary for the text, the opportunity is taken to use it to give an "elementary" proof of Theorem 3.1.

The first stage in the proof is to consider an arbitrary $f \in C[0,1]$

and the associated BVP of §2. The solution $\phi(\cdot,\cdot)$ which was produced gave rise to a semi-group $\{T_t | t \geq 0\}$ on $C[0,1]$, as we showed, where $(T_t f)(x) = \phi(x,t)$ for $t > 0$, all x. Fix t and observe that the functional $f \to (T_t f)(x)$ is positive with norm bounded by 1. Since $T_t 1 = 1$, the norm of the functional is exactly one.

It follows from the Riesz representation theorem that f is represented by a unique probability measure denoted $P_t(x,\cdot)$, in the sense that

$$(T_t f)(x) = \int f(y) P_t(x,dy).$$

We assert that $P_t(x,\cdot)$ restricted to $(0,1)$ is an absolutely continuous measure. To see this use Theorem 2.1 to show that

$$||T_t f||_2 \leq ||f||_2 \quad \text{for} \quad f \in C_0[0,1].$$

If the measure were singular, T_t could not even be bounded.

3.2 Lemma Suppose $0 \leq \alpha < \beta < \gamma < \delta \leq 1$. Then there exists a function $f \in C^1[0,1]$ satisfying:

(i) $f' \in AC[0,1]$;

(ii) $a(\cdot)f''(\cdot)$ is equivalent to a function in $C_0[0,1]$;

(iii) $f(x) = 1, \quad x \in [\beta,\gamma]$;

$f(x) = 0, \quad x \in [0,\alpha] \cup [\delta,1]$;

(iv) $0 \leq f(x) \leq 1$ all $x \in [0,1]$.

Proof: It suffices to show that there exists for any $\alpha < \beta$ a function f satisfying (i), (ii), together with:

(v) $f(\alpha) = 0 = f'(\alpha)$;

(vi) $f'(\beta) = 0, \quad f(\beta) = 1.$

Choose a continuous function h supported on $[\alpha+\varepsilon,\beta-\varepsilon]$ satisfying:

$$h(x) \geq 0 \qquad x \leq \frac{1}{2}(\alpha+\beta);$$

$$h(x) \leq 0 \qquad x \geq \frac{1}{2}(\alpha+\beta);$$

$$\int_0^1 h(x)a(x)^{-1}dx = 0.$$

Write
$$g(x) = \int_0^x h(u)a(u)^{-1}du,$$
$$f(x) = \int_0^x g(u)du.$$

It is easily seen that $a(\cdot)f''(\cdot) = h(\cdot)$ a.e., and all the remaining properties can be made to hold by multiplying f by some constant. □

Incidentally, this is the lemma needed to prove D_2 is uniformly dense $C_0[0,1]$, hence also dense in L_a^2.

3.3 Lemma Suppose $a(\cdot)f''(\cdot)$ is equivalent to some function $h \in C[0,1]$. Then

$$\lim_{t\to 0^+} \frac{1}{t}\{\int f(y)P_t(x,dy)-f(x)\} = h(x),$$

uniformly for $x \in [0,1]$.

Proof: Write $\phi(x,t) = \int f(y)P_t(x,dy)$,

$$\psi(x,t) = \int h(y)P_t(x,dy), \quad t \geq 0.$$

By definition of $P_t(x,dy)$ these are solutions to BVPs with starting functions f, h respectively. If both f and h vanish at 0,1, we may apply Corollary 1.5 to the eigenfunction expansions given in Theorem 2.1, deducing that $A_2\phi(\cdot,t) = \psi(\cdot,t)$, $t > 0$. (If f, h do not vanish at 0, 1, we appeal to the last paragraph of §2 .) . Then:

$$t^{-1}\{\phi(x,t)-\phi(x,0)\} = t^{-1}\int_0^t \frac{\partial\phi}{\partial s}(x,s)ds$$

$$= t^{-1}\int_0^t A_2\phi(x,s)ds \quad \text{a.e.}$$

$$= t^{-1}\int_0^t \psi(x,s)ds. \quad \text{a.e.}$$

But both sides are continuous in x, hence equal everywhere, and then Theorem 2.1 implies that the right hand side tends uniformly to $\psi(x,0)$, as $t \to 0^+$. Since $\psi(x,0) = h(x)$, we have finished the proof. □

3.4 Lemma Condition (o) of Theorem 3.1 is satisfied by $P_t(\cdot,\cdot)$.

Proof: Apply Lemma 3.3 to a function f satisfying the conditions of Lemma 3.2, noting that $(A_2f)(x_o) = 0$ for any point $x_o \in (\alpha,\beta)$. □

3.5 Lemma Let $f \in C^2[0,1]$. Then

$$\lim_{t\to 0^+} t^{-1}\{\int f(y)P_t(x,dy)-f(x)\} = a(x)f''(x)$$

at all points x where $a(\cdot)$ is continuous.

Proof: As in the proof of Lemma 3.3, write

$$t^{-1}\{\int f(y)P_t(x,dy)-f(x)\} = t^{-1}\int_0^t \psi(x,s)ds, \quad t > 0,$$

almost all x.

This reduces proof to showing that if $a(\cdot)$ is continuous at x, then

$$\lim_{s\to 0^+} \psi(x,s) = a(x)f''(x).$$

In the notation of Lemma 3.3, this means proving:

$$\lim_{t\to 0^+} \int h(y)P_t(x,dy) = h(x).$$

Proof will be finished by outlining a standard "pointwise convergence" argument which applies here. Approximate h above and below on $(x-\delta,x+\delta)$ by constant functions h_1, h_2. Use condition (o) (which we just proved in Lemma 3.4) to ignore integration outside $(x-\delta,x+\delta)$. Next show that $\limsup_{t\to 0^+} \int h(y)P_t(x,dy)$ may be estimated above using h_1, and that the lim inf is estimated below using h_2, finally concluding that the difference is arbitrarily small, hence zero. This proves $\lim_{t\to 0^+} \int h(y)P_t(x,dy) = h(x)$, as required. □

Proof of Theorem 3.1: Properties (iii), (iv) were established in §2, and (o) was proved in 3.4. For the case $\delta = 1$, condition (i) now follow from Lemma 3.5 by setting $f(y) = y$, and noting $f''(y) = 0$. If x_o is a point of continuity of $a(\cdot)$, write $f(y) = (y-x_o)^2$, so $a(x_o)f''(x_o) = 2a(x_o)$, and then Lemma 3.5 applies to f at x_o yeilding (ii).

To pass from the case $\delta = 1$ to general $\delta > 0$, use condition (o). $\square$